# MULTI-SCALE MODELING OF FUNCTIONALLY GRADED MATERIALS (FGMs) USING FINITE ELEMENT METHODS

by

Richard Sangwon Rhee

A Dissertation Presentation to the FACULTY OF THE GRADUATE SCHOOL UNIVERSITY OF SOUTHERN CALIFORNIA In Partial Fulfillment of the Requirements for the Degree DOCTOR OF PHILOSOPHY (CIVIL ENGINEERING)

December 2007

Richard Sangwon Rhee

Copyright 2007

# Dedication

This dissertation is dedicated to:

My parents who gave me everything and now they are in heaven;

M. Kim, who provides tremendous support;

My two hope, Ryan and Kyle;

Terry, who sacrifices during my education pursuits;

My elder sisters and brother-in-law, Jin Y. Sohn;

And J. Jackman, who gave me wisdom and encouragement.

## Acknowledgements

I would like to express my sincere thanks and gratitude to my advisor, Prof. L. C. Wellford, for giving me a chance to pursuit the Ph.D. program at Civil Engineering department and during academic years, he always shared his time in order to provide the information which I need. Also, I would like to thank to my dissertation committee members, Prof. V. Lee, Prof. S. Masri, Prof. J. Anderson and Prof. Shiflett for their valuable time and suggestions.

Sincere thanks to Kem Schankereli and Keith Myers, who encouraged me from the beginning of my graduation study. Also special thanks to my friends, Dr. H. J. Kwon and Dr. S. Kulkani, for their guidance. Last but not least, I would like to thank to my sworn brothers, Prof. S. J. Lee, Prof. K. D. Park and I. K. Jung, who always make me to believe my dream.

# **Table of Contents**

Dedication	ii
Acknowledgements	iii
List of Tables	vii
List of Figures	ix
Abstract	xiv
Chapter 1: Introduction	1
Chapter 2: Problems Involving a Micro Structure	
2.1 Review of Homogenization Theory	9
2.2 Analytical Solution of the Homogenization in 2-D	18
2.3 Macro Homogenized Solution	28
2.4 Variation of Cell Solution $\chi_i^{kl}$ in Microcell Geometries	30
2.4.1 A Typical Shape of Microcell Geometry 2.4.2 Computation of the Cell Solution $\chi_i^{kl}$ in Soft and	30
Hard Microcell Structure Cases	31
2.5 Verification of the Homogenized Elastic Constants	37
2.5.1 Soft and Hard Isotropic Composite Material with	
Variation	37
2.5.2 A Rectangular Hole in the Material	40
Chapter 3: Nonperiodic Homogenization (NPH) Method	
3.1 Mathematical Theory of NPH	43
3.2 Evaluation of the NPH Strain Tensors	47
3.3 Explicit Parameterization of the Cell Geometry over the	
Element	53
3.4 Coordinate Transformation	55
3.5 NPH Element Stiffness Matrix	58
3.6 NPH Element Stress Calculation	62
3.7 Typical Allowable NPH Microcell Geometries	64
3.8 Data Specification points for $hx_2 \& hy_2$ Matrix Values in	
the Nonperiodic Microstructure	66

Chapter 4: Numerical Examples of NPH 1-D Cases	
4.1 Introduction	68
4.2 Case 1 – Comparison between the NPH and the Homogenization	
Solutions	69
4.2.1 Geometry Modeling	69
4.2.2 Local and Global Deformation Analysis	70
4.3 Case 2 – Descending Low Density Microcell Structure	72
4.3.1 Geometry Modeling	73
4.3.2 Local and Global Deformation Analysis	73
4.4 Case 3 – Descending High Density Microcell Structure	75
4.4.1 Geometry Modeling	75
4.4.2 Local and Global Deformation Analysis	76
4.5 Case 4 – Descending and Ascending Microcell Structure	79
4.5.1 Geometry Modeling	79
4.5.2 Local and Global Deformation Analysis	80
4.6 Case 5 – Descending Microcell Structure with a Sudden Jump	82
4.6.1 Geometry Modeling	82
4.6.2 Local and Global Deformation Analysis	83
4.7 Case 6 – Rapidly Varying Descending, Ascending and	
Descending Microcell Structures	85
4.7.1 Geometry Modeling	85
4.7.2 Local and Global Deformation Analysis	86
Chapter 5: Numerical Examples of NPH 2-D Cases	0.0
5.1 Introduction	88
5.2 Case 7 – Periodic Microstructure	88
5.2.1 Geometry Modeling	88
5.2.2 Local and Global Deformation Analysis	90
5.3 Case 8 – Descending Horizontal Fiber Strips in One Direction	92
5.3.1 Geometry Modeling	92
5.3.2 Local and Global Deformation Analysis	94
5.4 Case 9 – Descending and Ascending FGMs with Square Fibers	96
5.4.1 Geometry Modeling	96
5.4.2 Local and Global Deformation Analysis	98
5.5 Case 10 – Descending and Symmetric Matrix Structure	101
5.5.1 Geometry Modeling	101
5.5.2 Local and Global Deformation Analysis	104
5.5.3 Local Stress Analysis	106
5.6 Case 11– Descending Horizontal and Vertical Fiber Strips	110
5.6.1 Geometry Modeling	110
5.6.2 Local and Global Deformation Analysis	113
5.6.3 Local Stress Analysis	115

Chapter 6: Summary and Conclusions	
6.1 Summary	118
6.2 Conclusions	121
Bibliography	123
Appendix A Nonperiodic Homogenization (NPH) Cell Solution	126

# List of Tables

Table 2.1	Comparison of the Homogenized Elastic Constants: Soft and Hard Material	39
Table 2.2	Elastic Tensors after Homogenization in Rectangular Hole	42
Table 4.1	Deformation Results of NPH and Homogenization Methods	71
Table 4.2	Deformation Results for the Descending Low Density Microcell Structure Case	74
Table 4.3	The Matrix Size Values for the Descending and Ascending Microcell Structure: Fiber size $hx_1 = 0.0262$	76
Table 4.4	The Deformation Results of the High Density Microcell Structure with 10 percent Reduction	77
Table 4.5	The Matrix Size Values for the Descending and Ascending Microcell Structure: Fiber size $hx_1 = 0.0262$	80
Table 4.6	Global Deformation Values in High Density Microcell Structure Case with 10 percent Descending and 16 percent Ascending Matrix Sizes	80
Table 4.7	The Matrix Size Values for the Descending Microcell Structure with a Sudden Jump: Fiber size $hx_1 = 0.0194$	83
Table 4.8	Table 4.8 Deformation Results for the High Density Descending Microcell Structure with a Sudden Jump	84
Table 4.9	Table 4.9 The Matrix Size Values for the Rapidly Varying Descending, Ascending and Descending Structure: Fiber size $hx_1 = 0.070$	86
Table 4.10	Deformation Results of the High Density Microcell with Rapidly Varying Descending, Ascending and Descending Microcell Structure	86
Table 5.1	Comparison of the Deformation between Homogenized Solution and NPH Solution @ $X_1$ =1.0, 1.5 & 2.0	91
Table 5.2	Matrix Size Values for the Descending Horizontal Fiber Strips	93

Table 5.3	Summary of Model Sizes for Case 8	94
Table 5.4	Matrix Sizes in Descending and Ascending Square Fibers	97
Table 5.5	Summary of Model Sizes for Case 9	98
Table 5.6	Matrix Sizes in Symmetric and Descending Case	103
Table 5.7	Summary of Model Sizes for Case 10	104
Table 5.8	Von-Mises Stress in Descending and Symmetric Case	109
Table 5.9	Matrix Sizes in Descending Horizontal and Vertical Strips Case	112
Table 5.10	Summary of Model Sizes for Case 11	113
Table 5.11	Von-Mises Stress in Horizontal and Vertical Strips	116

# **List of Figures**

Figure 1.1	Fiber-Reinforced Polymer Bridge	2
Figure 1.2	Reinforcement types of Metal Matrix Composites (MMCs) or Ceramics Matrix Composites (CMCs); a) Matrix with Fibers b) whiskers c) particulates	2
Figure 1.3	Different types of the Functionally Graded Materials (FGMs): a) Continuously Graded Microstructure; b) Discretely Graded Microstructure with fiber and matrix configuration; c) Multi- phase Graded Microstructures.	2
Figure 1.4	Replacement scheme used in the layer-wise Homogenization model in Continuously Graded Microstructure	4
Figure 1.5	Equivalent continuum with layers wise Homogenization in Discretely Graded Microstructure with fiber and matrix configuration (MMCs / CMCs types)	6
Figure 2.1	Periodic Microstructure	10
Figure 2.2	Associated Macrostructure	10
Figure 2.3	Base cell of the Composite Microstructure	10
Figure 2.4	Microcell Structure: Model 1 – Cell Size 1.0 x 1.0; Model 2 – Cell size 0.5 x 1.0	30
Figure 2.5	Microcell Solution for Model 1 – Cell size 1.0 x 1.0	31
Figure 2.6	Microcell Solution for Model 2 – Cell size 0.5 x 1.0	32
Figure 2.7	Superimpose the Cell Solution Pictures of Figure 2.5 and Figure 2.6	32
Figure 2.8	Microcell Solution for Model 1 – Cell size 1.0 x 1.0	33
Figure 2.9	Microcell Solution for Model 2 – Cell size 0.5 x 1.0	33
Figure 2.10	Superimpose the Cell Solution Pictures of Figure 2.8 and Figure 2.9	34

Figure 2.11	Microcell Solution for Model 1 – Cell Size 1.0 x 1.0	35
Figure 2.12	Microcell Solution for Model 2 – Cell Size 0.5 x 1.0	35
Figure 2.13	Superimpose the Cell Solution Pictures of Figure 2.11 and Figure 2.12	36
Figure 2.14	Material Properties in a Single Microcell Geometry: ( $E_{soft} = 10$ and $E_{hard} = 1000$ )	37
Figure 2.15	A Single Microcell Geometry Model Defined by Bendoe and Kikuchi [3]: 16 x 16 Uniform Square Four-node Isoparametric Elements	38
Figure 2.16	A Single Microcell Geometry Model Defined by HOMOG.f90: 8 x 8 Uniform Square Nine-node Lagrange Quadratic Elements	39
Figure 2.17	Material Properties in a Single Microcell Geometry with a Hole: $(E_{1111} = E_{2222} = 30 \text{ and } E_{1122} = E_{1212} = 10)$	40
Figure 2.18	A Single Microcell Geometry Model Defined by Bendoe and Kikuchi [3]: 20 x 20 Uniform Square Four-node Isoparametric Element	41
Figure 2.19	A Single Microcell Geometry Model Defined by HOMOG.f90: 10 x 10 Uniform Square Nine-node Lagrange Quadratic Elements	41
Figure 3.1	Nodal Degree-of-Freedom in 1-D Element	45
Figure 3.2	Nodal Displacement at the Mesh Element and Microstructure: i.e.) $E_1 = 10$ and $E_2 = 1000$	63
Figure 3.3	A Microcell Structure in 2-D (Shaded areas Represent Fibers, $hx_1$ and $hy_1$ )	64
Figure 3.4	Variation of 2-D Microcell Structures with Fiber (shaded area): (a) Vertical Direction; (b) Center; (c) Outer Edges; (d) Distributed in the Center and Four Corners	65
Figure 3.5	A Total of Three $hx_2$ Data Collection Points per One Finite Element: Black Dots indicate Data Collection Points and Circles with Cross indicate the Location of Gauss Points.	66

Figure 3.6	A Total of Nine Data Collection Points per Finite Element: Black Dots indicate Data Collection Points and Circles with Crosses indicate the Location of Gauss Points.	67
Figure 4.1	A Periodic Structure with Microcells	69
Figure 4.2	A Single Macro Element for the NPH Method	70
Figure 4.3	Detailed View of the Cell 1 for the Exact Solutions: $hx_2 = 0.375$ and $hx_1 = 0.250$	70
Figure 4.4	Three Different $hx_2$ Data Collection Methods; Black Dots indicate Data Collection Points and Circles with Cross indicate the Location of Gauss Points.	72
Figure 4.5	Nonperiodic Low Density Microcell Structure	73
Figure 4.6	Deformation Results for the Descending Low Density Microcell with Three Different Data Collection Methods	74
Figure 4.7	Descending Microcell Structure Case with 10 percent Descending	75
Figure 4.8	Deformation Results of Descending Microcell Structure in High Density Microcell with Three Different Data Collection Methods	78
Figure 4.9	Detailed View of the Figure 4.5 (Red Box Region)	78
Figure 4.10	High Density Microcell Structure Case with 10 percent Descending and 16 percent Ascending Matrix Sizes	79
Figure 4.11	Deformation Results for High Density Descending and Ascending Microcell Structure	81
Figure 4.12	Detailed View of the Figure 4.5 (Red Box Region)	81
Figure 4.13	Descending Micro Structure with a Sudden Jump	82
Figure 4.14	Deformation Results of the High Density Descending Micro Structure with a Sudden Jump	84
Figure 4.15	High Density Microcells with Rapidly Varying Descending, Ascending and Descending Structure	85

Figure 4.16	Deformation Results of the High Density Microcell with Rapidly Varying Descending, Ascending and Descending Structure	87
Figure 5.1	Periodic Microcell Geometry and Boundary Conditions	89
Figure 5.2	Homogenization and NPH Deformation Results for the Periodic Microstructure at $X_1 = 1.0$ , $X_1 = 1.5$ and $X_1 = 2.0$	90
Figure 5.3	Descending Horizontal Fiber Strips Microcell Structure: (a) Descending Horizontal Fiber Strips in X <sub>2</sub> Direction, (b) 16 Macro Elements for NPH	92
Figure 5.4	Detailed View of the Microcell Strip: Fiber Value $hy_1 = 0.030$ (Constant)	93
Figure 5.5	Conventional FEA Mesh and Boundary Conditions	94
Figure 5.6	Local Deformation Results: the NPH Solution and the FEA Solution at $X_1 = 1.995$	95
Figure 5.7	Descending and Ascending Matrix with Square Fibers	96
Figure 5.8	Detailed View of the Square Fibers: Fiber Value $hx_1 \& hy_1 = 0.030$ (Constant)	97
Figure 5.9	Conventional FEA Mesh Sizes and Boundary Conditions	99
Figure 5.10	Detailed View of the FEA Mesh Sizes and Boundary Conditions (Red Box Region)	99
Figure 5.11	Deformation Results Using the Conventional FEA Method	99
Figure 5.12	Deformation Results of the Conventional FEA Method and the NPH Method at $X_1 = 1.9427$	100
Figure 5.13	Descending Matrix Microstructures in $X_1$ Direction and Symmetric condition in $X_2$ Direction with Descending Microstructures	101
Figure 5.14	Detailed View of Figure 5.13 (Red Box Region)	102
Figure 5.15	Symmetric and Descending Structure: Deformation Displacement at $X_1 = 1.995$	105

Figure 5.16	Detailed View of Symmetric and Descending Structure (Red Box Region)	105
Figure 5.17	Deformation Results of the FEA Method (ABAQUS)	106
Figure 5.18	Von-Mises Stress Results from FEA (ABAQUS)	107
Figure 5.19	Detailed View of the High Stress Region (Red Box Region): Max. Stress = 5.811	107
Figure 5.20	Von-Mises Stress Results by NPH method (Red Box Region in Figure 5.19): Max. Stress Value = 7.473	108
Figure 5.21	Von-Mises Stress Results by Commercially Available FEA (ABAQUS): Max. Stress Value = 5.777	109
Figure 5.22	Simultaneously Reducing Matrix Values in X1 & X2 Directions	110
Figure 5.23	Detailed View of the Microcell Structure	111
Figure 5.24	Deformation Results of Descending Horizontal and Vertical Strips	114
Figure 5.25	FEA (ABAQUS) Deformation Result	114
Figure 5.26	Von-Mises Stress Results from FEA (ABAQUS)	115
Figure 5.27	Detailed View of Figure 5.26 (Red Box Region): Maximum Stress Value is 11.35	116
Figure 5.28	Von-Mises Stress Results by the NPH Method at Integration Points: Maximum Stress = 13.86	117
Figure 5.29	Von-Mises Stress Results by FEA (ABAQUS) at Integration Points: Maximum Stress = 12.11	117

### Abstract

Functionally Graded Materials (FGMs) have a gradual material variation from one material character to another throughout the structure. Applications of these types of materials have significant advantages in civil and mechanical systems including thermal systems. Analyzing the FGMs at the microstructure level with the conventional Finite Element Method (FEM) takes enormous pre-processing and computational time due to the complex material characteristics at the microstructure level. Essentially, the model contains too many degrees of freedom to be solved economically.

The homogenization method has been successfully applied to solve periodic microstructure problems. However, the development of analysis procedures for structures with nonperiodic material or cell geometry, as occurs in graded materials, has turned out to be a significant challenge.

A new method is developed which accurately models the nonperiodic microstructure in FGMs. This method allows the efficient solution of nonperiodic problems without requiring the simplification of the original models. The performance of the developed theory is verified through the solution of appropriate nonperiodic problems associated with graded materials. In the nonperiodic 1-D cases, the global displacement  $U(\mathbf{x})$  was obtained and compared with the exact solution. At the same time, the proposed data collection point method was investigated. In the nonperiodic 2-D cases, the global displacement  $U(\mathbf{x})$  and the microstructural level

xiv

displacements were computed. In the program, the Von-Mises Stress computation process was included to evaluate the local stress values at the microstructure level and the results were compared with very fine scale finite element calculations.

The performance of the developed nonperiodic homogenized (NPH) algorithm indicates that it is a promising tool for estimating the FGMs characteristics in loaded conditions. The method can be applied to estimate the global and local displacements in nonperiodic geometries which contain continuously decreasing and/or increasing microstructures.

## Chapter 1

#### Introduction

Functionally Graded Materials (FGMs) are composite materials which have a gradual variation of the volume fractions from one material to the other. The material property changes are usually in one direction with two different materials being used. The microstructure is organized in a nonperiodic manner. The concept of FGMs originated in Japan during the space plane project in 1984. The Japanese scientists were developing a thermal barrier material which could withstand a surface temperature gradient of approximately 1000 K across a 10 mm cross section. Traditional thermal barrier coatings (TBCs) have been applied with Ni-based alloys as the oxidation resistant bond coat and a heat resistant  $ZrO_2$  ceramic top coating. However, conventional plasma sprayed TBCs have a problem of low durability during thermal cycling and poor bond strength. The uniqueness of the FGMs is the ability to produce a new composite material with a gradual composition variation from heat resistant ceramics to fracture resistant metals. Applications of these types of FGMs have significant advantages in civil and mechanical systems including thermal systems (e.g. rocket heat shields, heat exchanger tubes, thermoelectric generators, wearresistant linings, diesel and turbine engines, etc.).

Fiber-reinforced polymer (FRP) is another application of FGMs for reinforcing concrete materials as shown in Figure 1.1. The FRP materials improve the corrosion resistance of the steel and enhance the life cycle of the material strength.



Figure 1.1 Fiber-Reinforced Polymer Bridge



Figure 1.2 Reinforcement types of Metal Matrix Composites (MMCs) or Ceramics Matrix Composites (CMCs); a) Matrix with Fibers b) whiskers c) particulates



Figure 1.3 Different types of the Functionally Graded Materials (FGMs): a) Continuously Graded Microstructure; b) Discretely Graded Microstructure with fiber and matrix configuration; c) Multi-phase Graded Microstructures.

FRP has been used in several bridge decks recently constructed in North America; The Morristown Bridge, in Vermont, has an entire concrete deck slab constructed using glass FRP (GFRP) bars. Also, there are laminate types of FRP which are contained in an arrangement of unidirectional fibers or woven fiber fabrics embedded in a thin layer of light polymer matrix materials - polyester, Epoxy or Nylon, etc.

Other major benefits of the FGMs are in the design and manufacturing of Metal Matrix Composites (MMCs) and Ceramic Matrix Composites (CMCs). The associated manufacturing processes provide the best material properties for composites of metal and ceramics by, for examples, removing the brittleness of ceramics and making a strong metal lighter and stiffer. The proportions of the matrix alloy (the metal) and the reinforcement material (the ceramic), as well as shape and location of reinforcement can be varied form place to place in the structure to achieve particular desired properties. For example, ceramic reinforcements in the form of fibers, whiskers or particulates can be introduced into the metal in a varying density pattern as shown in Figure 1.2. However, the use of this type of material requires an explicit understanding of the material behavior at each location and over all length of scales.

Many more applications of the FGMs can be found in the conference papers and technical journals including those related to solar energy conversion devices [20]. Most of the FGMs microstructures are fabricated in three major types: continuously graded microstructure, discretely graded microstructure, and multi-phase graded microstructure as pictured in Figure 1.3.

3

Analyzing the FGMs at the microstructure level with conventional Finite Element Method (FEM) takes enormous pre-process and computational time. Essentially, the model contains too many degrees of freedom to solve economically. Therefore, many researchers have tried to analyze the FGMs using various methods, which increase the accuracy of numerical solution and enhance the prediction of local stress concentrations.

One of the common methods is to divide the FGMs domain into multi-layers in the direction of the material gradation and to apply the traditional homogenization method within each layer [27, 31] as shown in Figure 1.4. These layer-wise averaged models are based on the self-consistent method [15]. In order to minimize the errors in the layer-wise homogeneous model, Vemaganti and Deshmukh [34] used the adaptive approach to model the FGMs. Some of the researchers, such as Sandra and Lambros [29] varied the material property matrix in the finite element with a simple boundary condition. Kim and Paulino [19] used an exponential variation of material elastic properties to model nonhomogeneous, isotropic and orthotropic, materials.



Figure 1.4 Replacement scheme used in the layer-wise Homogenization model in Continuously Graded Microstructure

Higher order theory was proposed by Aboudi, et. al. [1, 25]. This theory allows the thermo-inelastic analysis of materials with spatially varying microstructure based on volumetric averaging of the various field quantities and satisfaction of the field equations. The use of the finite-element discretization approach utilizing rectangular cells was compared with the averaging estimation methods in linear, modified rules of mixtures (average Young's modulus) and the Wakashima-Tsukamoto estimate presented by Cho and Ha [7]. Also, Berezovski A. et al. [4] used the linear rule of mixtures to define the material properties in his study of wave propagation in FGMs. On the other hand, a discrete micromechanics approach, using planar hexagonal cells of equal size, was developed by Ghosh et al. [10].

At the case of the MMCs or CMCs, layers-wise process also applied to replace the heterogeneous microstructure properties to an equivalent continuum with a set of macroscopic properties as shown in Figure 1.5. However, in this case no coupling exists between local and global responses and these properties are calculated without taking into account of the influence of the adjacent variable micro structural details explicitly.

Modeling of material with microstructure has been carried out using classical asymptotic homogeneous methods. In the classic methods, the microstructure is periodic and is associated with a microstructure cell. However, the material distribution of the FGMs, because of the definition of micro structural properties, is arbitrary. The micro structure is not periodic in the length scale of the macrostructure.

5



Figure 1.5 Equivalent continuum with layers wise Homogenization in Discretely Graded Microstructure with fiber and matrix configuration (MMCs / CMCs types)

This leaves a major difficulty because very few analysis methods exist to solve a structural problem which has a nonperiodic microstructure.

Therefore, in this proposal, a new theory of coupling the micro-macro structural models is developed. The new method is capable of dealing with the nonperiodic microstructure of the FGMs. A nonperiodic homogenization (NPH) theory is created to analyze the FGMs models. This model will handle microstructure with fiber/matrix combinations. The nonperiodic homogenization algorithm has been developed to solve the problems with nonperiodic, arbitrarily spaced inclusions or continuous fibers composite materials. The developed algorithm links the new microstructural model with conventional Finite Element Method (FEM) discredited technique.

Chapter 2 reviews the fundamental homogenization theory and defines the homogenized elastic constant. Based on the microstructure cell solution and

homogenized elastic constant, 2-D cases with microstructure were solved, and the solution was compare to analytical solutions. These results will be used for creating the HOMOG algorithm. The algorithm was verified with the results of two different examples, which were demonstrated from M. P. Bendsoe and N. Kikuchi [3].

Chapter 3 presents the nonperiodic homogenization theory and mathematical formulations which are transferred from the Cartesian coordinate system to the natural coordinate system. The details of conversion processes are shown in Appendix B.1. Based on the mathematical formulations, the finite element stiffness matrix has been defined and used in the creation of the nonperiodic homogenization algorithm. Many variations of generic microcell structure were shown to fix frequently used FGMs geometries.

Chapter 4 presents the results of 1-D cases of the nonperiodic homogenization program. Most of the verifications are conducted in 1-D cases in order to compare with the available analytical solutions. The model problems and associated boundary conditions are described. The verification cases are: Case 1 – Comparison between the NPH and the Homogenization Solution, Case 2 – Descending Low Density Microcell Structure, Case 3 – Descending High Density Microcell Structure, Case 4 – Descending and Ascending Microcell Structure, Case 5 – Descending Microcell Structure with a Sudden Jump and Case 6 – Rapidly Varying Descending, Ascending and Descending Microcell Structures.

Chapter 5 presents the 2-D verification cases for FGMs. The verification cases are Case 7 – Periodic Microstructure, Case 8 – Descending Horizontal Fiber Strips in

7

One Direction, Case 9 – Descending and Ascending FGMs with Square Fibers, Case 10 – Descending and Symmetric Matrix Structure and Case 11 – Descending Horizontal and Vertical Fiber Strips. Von-Mises stresses are presented for the Case 10 and Case 11.

Finally, Appendix A presents the nonperiodic homogenization coordinate transformation from the Cartesian coordinate system to the natural coordinate system.

## Chapter 2

## Problems Involving a Microstructure

#### 2.1 Review of Homogenization Theory

The following analysis represents the homogenization theory for the periodic microstructural case. This is the starting point for the analysis of the nonperiodic microstructure.

Consider the 2-D case. A solid body is made of two different materials and has a base cell geometry of order  $\varepsilon$  (very small positive number) in size as shown in Figure 2.1. Suppose the material properties in the base cell vary rapidly from point to point producing heterogeneity. Thus, it is reasonable to assume that all quantities have two explicit dependences. One is on the macroscopic level **x** coordinate, and the other one is on the microscopic level coordinate  $\mathbf{x}/\varepsilon$ . If g is a general function,  $g = g(\mathbf{x}, \mathbf{x}/\varepsilon)$  and  $\mathbf{x}/\varepsilon$  can be replaced with  $\mathbf{y} (= \mathbf{x}/\varepsilon)$ . Due to the periodic nature of the microstructure, the dependence of a function on the micro-coordinate  $\mathbf{y}$  is periodic. The quantity  $1/\varepsilon$  can be thought of as a magnification factor, which enlarges the dimensions of a base cell,  $\mathbf{y} = \mathbf{x}/\varepsilon$ . Let  $\Omega$  be an open subset of  $\mathbb{R}^2$  with a smooth, boundary  $\Gamma$  as described in Figure 2.2. The Figure 2.2 depicts the associated macrostructure.



Figure 2.1 Periodic Microstructure



Figure 2.2 Associated Macrostructure



Figure 2.3 Base cell of the Composite Microstructure

Let Y be a rectangular region in two dimensional spaces defined by

$$\mathbf{Y} = (0, y_1^0) \, \mathbf{X}(0, y_2^0) \,, \tag{2.1}$$

Let  $\vartheta$  be an open subset of **Y** with boundary

$$\partial \rho = S \tag{2.2}$$

and let

$$\pi = \overline{Y}/\overline{\rho} , \qquad (2.3)$$

where  $\pi$  is the solid part of the cell with a material property  $E_1$ ,  $\overline{\rho}$  denotes the closure of  $\rho$  and **Y** represents the base cell of the composite microstructure. The base cell properties vary inside **Y**, and the set  $\rho$  represents a material property  $E_2$  inside  $\overline{Y}$ . Define now,

$$\overline{\Theta}(\overline{y}) = \begin{cases} E_1 & \text{if } \overline{y} \in \pi, \\ E_2 & \text{if } \overline{y} \notin \pi, \end{cases}$$
(2.4)

and extend  $\overline{\Theta}$  to  $\mathbb{R}^2$  by  $\varepsilon$  periodicity (i.e. it repeats the base cell in all two direction). The superscript  $\varepsilon$  is the characteristic inhomogeneity dimension. Find  $\mathbf{u} \in \mathbf{V}^{\varepsilon}$ , such that

$$\int_{\Omega^{\varepsilon}} E_{ijkl}^{\varepsilon} \frac{\partial u_k^{\varepsilon}}{\partial x_l} \frac{\partial v_i}{\partial x_j} d\Omega = \int_{\Omega^{\varepsilon}} f_i^{\varepsilon} v_i d\Omega + \int_{\Gamma_l} t_i v_i d\Gamma + \int_{\partial\Omega^{\varepsilon}} p_i^{\varepsilon} v_i dS \qquad \forall \mathbf{v} \in \mathbf{V}^{\varepsilon}$$
(2.5)

Here, it is assumed that the stress-strain and strain-displacement relations are

$$\sigma_{ij}^{\varepsilon} = E_{ijkl}^{\varepsilon} e_{kl}^{\varepsilon} , \qquad (2.6)$$

$$e_{kl}^{\varepsilon} = \frac{1}{2} \left( \frac{\partial u_k^{\varepsilon}}{\partial x_l} + \frac{\partial u_l^{\varepsilon}}{\partial x_k} \right), \tag{2.7}$$

and that the elastic constants have the following properties:

$$E_{ijkl}^{\varepsilon} = E_{jikl}^{\varepsilon} = E_{ijlk}^{\varepsilon} = E_{klij}^{\varepsilon}, \qquad (2.8)$$

Since the body forces  $\overline{f}$ , tractions  $\overline{p}$  and the elastic constants very with the small cells of the composite (i.e., they are functions of both **x** and **y** = **x**/ $\varepsilon$ ). The general functional form in the two scale approximation is,

$$\Phi^{\varepsilon}(\mathbf{x}) = \Phi(\mathbf{x}, \mathbf{y}), \quad \mathbf{y} = \mathbf{x}/\varepsilon, \qquad (2.9)$$

The displacement solution  $\mathbf{u}^{\varepsilon}$  take this form; that is,

$$\mathbf{u}^{\varepsilon}(\mathbf{x}) = u(\mathbf{x}, \mathbf{y}), \quad \mathbf{y} = \mathbf{x}/\varepsilon,$$
 (2.10)

Using a two scale asymptotic expansion,

$$\mathbf{u}^{\varepsilon}(\mathbf{x}) = \mathbf{u}^{0}(\mathbf{x}, \mathbf{y}) + \varepsilon \mathbf{u}^{1}(\mathbf{x}, \mathbf{y}) + \varepsilon^{2} \mathbf{u}^{2}(\mathbf{x}, \mathbf{y}) + \dots, \quad \mathbf{y} = \mathbf{x}/\varepsilon$$
(2.11)

where,

$$\mathbf{u}^{j}(\mathbf{x},\mathbf{y})$$
 is defined in  $(\mathbf{x},\mathbf{y}) \in \Omega \times \pi$ ,

 $\mathbf{y} \rightarrow \mathbf{u}^{j}(\mathbf{x}, \mathbf{y})$  is Y- periodic.

Fact 1: The derivative of a periodic function is also periodic.

Fact 2: The integral of the derivative of a function over the period is zero.

Fact 3: If  $\Phi = \Phi(x, y)$  and y depends on x, then

$$\frac{d\Phi}{dx} = \frac{\partial\Phi}{\partial x} + \frac{\partial\Phi}{\partial y}\frac{\partial y}{\partial x}, \qquad y = x/\varepsilon$$

$$\frac{d\Phi}{dx} = \frac{\partial\Phi}{\partial x} + \frac{1}{\varepsilon}\frac{\partial\Phi}{\partial y} \qquad (2.12)$$

Equation (2.5) becomes,

$$\int_{\Omega^{\varepsilon}} E_{ijkl}^{\varepsilon} \frac{\partial u_{k}^{\varepsilon}}{\partial x_{l}} \frac{\partial v_{i}}{\partial x_{j}} d\Omega = \int_{\Omega^{\varepsilon}} E_{ijkl}^{\varepsilon} \left[ \left( \frac{\partial u_{k}^{0}}{\partial x_{l}} + \frac{1}{\varepsilon} \frac{\partial u_{k}^{0}}{\partial y_{l}} \right) \left( \frac{\partial v_{i}}{\partial x_{j}} + \frac{1}{\varepsilon} \frac{\partial v_{i}}{\partial y_{j}} \right) \right] \\ + \varepsilon \left( \frac{\partial u_{k}^{1}}{\partial x_{l}} + \frac{1}{\varepsilon} \frac{\partial u_{k}^{1}}{\partial y_{l}} \right) \left( \frac{\partial v_{i}}{\partial x_{j}} + \frac{1}{\varepsilon} \frac{\partial v_{i}}{\partial y_{j}} \right) + \varepsilon^{2} (\dots) d\Omega$$

$$= \int_{\Omega^{\varepsilon}} E_{ijkl}^{\varepsilon} \left[ \frac{\partial u_{k}^{0}}{\partial x_{l}} \frac{\partial v_{i}}{\partial x_{j}} + \frac{1}{\varepsilon} \frac{\partial u_{k}^{0}}{\partial x_{l}} \frac{\partial v_{i}}{\partial y_{j}} + \frac{1}{\varepsilon} \frac{\partial u_{k}^{0}}{\partial y_{l}} \frac{\partial v_{i}}{\partial x_{j}} + \frac{1}{\varepsilon^{2}} \frac{\partial u_{k}^{0}}{\partial y_{l}} \frac{\partial v_{i}}{\partial y_{j}} \right]$$

$$+ \varepsilon \frac{\partial u_{k}^{1}}{\partial x_{l}} \frac{\partial v_{i}}{\partial x_{j}} + \frac{\partial u_{k}^{1}}{\partial x_{l}} \frac{\partial v_{i}}{\partial y_{j}} + \frac{\partial u_{k}^{1}}{\partial y_{l}} \frac{\partial v_{i}}{\partial x_{j}} + \frac{1}{\varepsilon} \frac{\partial u_{k}^{1}}{\partial y_{l}} \frac{\partial v_{i}}{\partial y_{j}} + \dots \right] d\Omega$$

$$(2.14)$$

Now, rearrange for the terms in  $\frac{1}{\varepsilon^2}$ ,  $\frac{1}{\varepsilon}$  and  $\varepsilon$  then,

$$= \int_{\Omega^{\varepsilon}} E_{ijkl}^{\varepsilon} \left\{ \frac{1}{\varepsilon^{2}} \frac{\partial u_{k}^{0}}{\partial y_{l}} \frac{\partial v_{i}}{\partial y_{j}} + \frac{1}{\varepsilon} \left[ \left( \frac{\partial u_{k}^{0}}{\partial x_{l}} + \frac{\partial u_{k}^{1}}{\partial y_{l}} \right) \frac{\partial v_{i}}{\partial y_{j}} + \frac{\partial u_{k}^{0}}{\partial y_{l}} \frac{\partial v_{i}}{\partial x_{j}} \right] \right. \\ \left. + \left[ \left( \frac{\partial u_{k}^{0}}{\partial x_{l}} + \frac{\partial u_{k}^{1}}{\partial y_{l}} \right) \frac{\partial v_{i}}{\partial x_{j}} + \left( \frac{\partial u_{k}^{1}}{\partial x_{l}} + \frac{\partial u_{k}^{2}}{\partial y_{l}} \right) \frac{\partial v_{i}}{\partial y_{j}} \right] + \varepsilon(\dots) \right\} d\Omega$$
(2.15)

Equation (2.14) becomes,

$$\int_{\Omega^{\varepsilon}} E_{ijkl}^{\varepsilon} \begin{cases} \frac{1}{\varepsilon^{2}} \frac{\partial u_{k}^{0}}{\partial y_{l}} \frac{\partial v_{i}}{\partial y_{j}} + \frac{1}{\varepsilon} \left[ \left( \frac{\partial u_{k}^{0}}{\partial x_{l}} + \frac{\partial u_{k}^{1}}{\partial y_{l}} \right) \frac{\partial v_{i}}{\partial y_{j}} + \frac{\partial u_{k}^{0}}{\partial y_{l}} \frac{\partial v_{i}}{\partial x_{j}} \right] \\ + \left[ \left( \frac{\partial u_{k}^{0}}{\partial x_{l}} + \frac{\partial u_{k}^{1}}{\partial y_{l}} \right) \frac{\partial v_{i}}{\partial x_{j}} + \left( \frac{\partial u_{k}^{1}}{\partial x_{l}} + \frac{\partial u_{k}^{2}}{\partial y_{l}} \right) \frac{\partial v_{i}}{\partial y_{j}} \right] + \varepsilon(...) \end{cases} d\Omega$$

$$= \int_{\Omega^{\varepsilon}} f_{i}^{\varepsilon} v_{i} d\Omega + \int_{\Gamma_{t}} t_{i} v_{i} d\Gamma + \int_{S^{\varepsilon}} p_{i}^{\varepsilon} v_{i} dS \qquad (2.16)$$

The functions are smooth enough so that the limit when  $\varepsilon \rightarrow 0$  of all integrals exist then, equation (2.16) holds if the terms of the same power of  $\varepsilon$  are equal to zero. Therefore,

$$\frac{1}{\varepsilon^{2}} \int_{\Omega^{\varepsilon}} E_{ijkl}^{\varepsilon} \frac{\partial u_{k}^{0}}{\partial y_{l}} \frac{\partial v_{i}}{\partial y_{j}} d\Omega = 0 \qquad \forall \mathbf{v} \in V_{\Omega \times \pi} \tag{2.17}$$

$$\frac{1}{\varepsilon} \int_{\Omega^{\varepsilon}} E_{ijkl}^{\varepsilon} \left[ \left( \frac{\partial u_{k}^{0}}{\partial x_{l}} + \frac{\partial u_{k}^{1}}{\partial y_{l}} \right) \frac{\partial v_{i}}{\partial y_{j}} + \frac{\partial u_{k}^{0}}{\partial y_{l}} \frac{\partial v_{i}}{\partial x_{j}} \right] d\Omega = \int_{S^{\varepsilon}} p_{i}^{\varepsilon} v_{i} dS \quad \forall \mathbf{v} \in V_{\Omega \times \pi} \tag{2.18}$$

$$\int_{\Omega^{\varepsilon}} E_{ijkl}^{\varepsilon} \left[ \left( \frac{\partial u_{k}^{0}}{\partial x_{l}} + \frac{\partial u_{k}^{1}}{\partial y_{l}} \right) \frac{\partial v_{i}}{\partial x_{j}} + \left( \frac{\partial u_{k}^{1}}{\partial x_{l}} + \frac{\partial u_{k}^{2}}{\partial y_{l}} \right) \frac{\partial v_{i}}{\partial y_{j}} \right] d\Omega$$

$$= \int_{\Omega^{\varepsilon}} f_{i}^{\varepsilon} v_{i} d\Omega + \int_{\Gamma_{t}} t_{i} v_{i} d\Gamma \qquad \forall \mathbf{v} \in V_{\Omega \times \pi} \tag{2.19}$$

Fact 4: 
$$\lim_{\varepsilon \to 0^+} \int_{\Omega^{\varepsilon}} \Psi\left(\frac{\mathbf{x}}{\varepsilon}\right) d\Omega \to \frac{1}{|Y|} \iint_{\Omega \pi} \Psi(\mathbf{y}) dY d\Omega,$$
  
$$\lim_{\varepsilon \to 0^+} \varepsilon \int_{S^{\varepsilon}} \Psi\left(\frac{\mathbf{x}}{\varepsilon}\right) d\Omega \to \frac{1}{|Y|} \iint_{\Omega S} \Psi(\mathbf{y}) dS d\Omega,$$

v is an arbitrary function, thus we choose v = v(y) then integrating by parts, applying the divergence thermo to the integral in  $\pi$ . Equations (2.17) thru (2.19) becomes,

$$\frac{1}{|Y|} \iint_{\Omega \pi} E_{ijkl} \frac{\partial u_k^0}{\partial y_l} \frac{\partial v_i}{\partial y_j} dY d\Omega = 0 \qquad \forall \mathbf{v} \in V_{\Omega \times \pi}$$
(2.20)  
$$\frac{1}{|Y|} \iint_{\Omega} \left\{ \iint_{\pi} \left[ -\frac{\partial}{\partial y_i} \left( E_{ijkl} \frac{\partial u_k^0}{\partial y_l} \right) \right] v_i dY + \iint_{S} E_{ijkl} \frac{\partial u_k^0}{\partial y_l} n_j v_i dS \right\} d\Omega = 0 \quad \forall \mathbf{v} \in V_{\Omega \times \pi}$$
(2.21)

From the equation (2.21) becomes,

$$-\frac{\partial}{\partial y_i} \left( E_{ijkl} \frac{\partial u_k^0}{\partial y_l} \right) = 0, \quad y \in \pi,$$
(2.22)

$$E_{ijkl} \frac{\partial u_k^0}{\partial y_l} n_j = 0 \text{ on } S$$
(2.23)

Thus, this indicates that the first term of the expansion equation is depends only on  $\mathbf{x}$ , which is the macroscopic scale.

$$\mathbf{u}^{0}(\mathbf{x},\mathbf{y}) = \mathbf{u}^{0}(\mathbf{x}) \tag{2.24}$$

Now inset equation (2.24),  $\mathbf{u}^0 = \mathbf{u}^0(\mathbf{x})$ , into equation (2.18) and multiplying by  $\varepsilon$  and using Fact (4). Since, v = v(y) which means it's derivative will be zero,

$$\frac{\partial v}{\partial x} = 0.$$

$$\frac{1}{\varepsilon} \int_{\Omega^{\varepsilon}} E_{ijkl}^{\varepsilon} \left[ \left( \frac{\partial u_{k}^{0}}{\partial x_{l}} + \frac{\partial u_{k}^{1}}{\partial y_{l}} \right) \frac{\partial v_{i}}{\partial y_{j}} + \frac{\partial u_{k}^{0}}{\partial y_{l}} \frac{\partial v_{i}}{\partial x_{j}} \right] d\Omega = \int_{S^{\varepsilon}} p_{i}^{\varepsilon} v_{i} dS \quad \forall \mathbf{v} \in V_{\Omega \times \pi} \quad (2.25)$$

$$\int_{\Omega^{\varepsilon}} E_{ijkl}^{\varepsilon} \left[ \left( \frac{\partial u_k^0}{\partial x_l} + \frac{\partial u_k^1}{\partial y_l} \right) \frac{\partial v_i}{\partial y_l} \right] d\Omega = \int_{\Omega} \left( \frac{1}{|Y|} \int_{S} p_i v_i dS \right) d\Omega$$
(2.26)

$$\int_{\Omega^{\varepsilon}} \frac{1}{|Y|} \int_{\pi} E_{ijkl}^{\varepsilon} \left( \frac{\partial u_k^0(\mathbf{x})}{\partial x_l} + \frac{\partial u_k^1}{\partial y_l} \right) \frac{\partial v_i(\mathbf{y})}{\partial y_j} dY d\Omega = \int_{\Omega^{\varepsilon}} \left( \frac{1}{|Y|} \int_{S} p_i v_i dS \right) d\Omega$$
(2.27)

From the equation (2.27),

$$\int_{\pi} E_{ijkl}^{\varepsilon} \left( \frac{\partial u_k^0(\mathbf{x})}{\partial x_l} + \frac{\partial u_k^1}{\partial y_l} \right) \frac{\partial v_i(\mathbf{y})}{\partial y_j} dY = \int_{S} p_i v_i dS \text{ is linear with respect to } \mathbf{u}^0 \text{ and } p_i.$$

Thus, if  $p_i = 0$ , then,

$$-\int_{\pi} E_{ijkl}^{\varepsilon} \frac{\partial u_k^0(\mathbf{x})}{\partial x_l} \frac{\partial v_i(\mathbf{y})}{\partial y_j} dY = \int_{\pi} E_{ijkl}^{\varepsilon} \frac{\partial u_k^1}{\partial y_l} \frac{\partial v_i(\mathbf{y})}{\partial y_j} dY$$
(2.28)

or,

$$\int_{\pi} E_{ijkl}^{\varepsilon} \frac{\partial u_k^1}{\partial y_l} \frac{\partial v_i(\mathbf{y})}{\partial y_j} dY = -\int_{\pi} E_{ijkl}^{\varepsilon} \frac{\partial u_k^0(\mathbf{x})}{\partial x_l} \frac{\partial v_i(\mathbf{y})}{\partial y_j} dY$$
(2.29)

Let 
$$u_p^1 = -\chi_p^{kl}(\mathbf{x}, \mathbf{y}) \frac{\partial u_k^0(\mathbf{x})}{\partial x_l}$$
 (2.30)

and 
$$\frac{\partial u_p^1}{\partial y_q} = -\frac{\partial \chi_p^{kl}(\mathbf{x}, \mathbf{y})}{dy_q} \frac{\partial u_k^0(\mathbf{x})}{\partial x_l}$$
 substitute to equation (2.29) then,

$$-\int_{\pi} E_{ijkl}^{\varepsilon} \frac{\partial \chi_{p}^{kl}(\mathbf{x}, \mathbf{y})}{dy_{q}} \frac{\partial u_{k}^{0}(\mathbf{x})}{\partial x_{l}} \frac{\partial v_{i}(\mathbf{y})}{\partial y_{j}} dY = -\int_{\pi} E_{ijkl}^{\varepsilon} \frac{\partial u_{k}^{0}(\mathbf{x})}{\partial x_{l}} \frac{\partial v_{i}(\mathbf{y})}{\partial y_{j}} dY \quad (2.31)$$

Since,  $\frac{\partial u_k^0(\mathbf{x})}{\partial x_l} = 1$ , the equation (2.31) becomes,

$$\int_{\pi} E_{ijkl}^{\varepsilon} \frac{\partial \chi_{p}^{kl}(\mathbf{x}, \mathbf{y})}{dy_{q}} \frac{\partial v_{i}(\mathbf{y})}{\partial y_{j}} dY = \int_{\pi} E_{ijkl}^{\varepsilon} \frac{\partial v_{i}(\mathbf{y})}{\partial y_{j}} dY \quad \forall \mathbf{v} \in \mathbf{V}_{\pi}$$
(2.32)

$$E_{ijkl}^{H}(\mathbf{x}) = \frac{1}{|Y|} \int_{\pi} \left( E_{ijkl} - E_{ijpm} \frac{\partial \chi_{p}^{kl}}{\partial y_{m}} \right) dY$$
(2.33)

 $E_{ijkl}^{H}$  defined by equation (2.33) represents the homogenized elastic constant.

# 2.2 Analytical Solution of the Homogenization in 2-D

For the 2-D problems, it would be sufficient to solve them for the cases;

```
Case A: k = l = 1
Case B: k = l = 2
Case C: k = 1, l = 2 (or k = 2, l = 1)
```

#### <u>Case A: k = l = 1</u>

Expand the equation (2.32) and remove terms with zero coefficients,

$$= \int_{\pi} \left( E_{1111} \frac{\partial \chi_{1}^{11}}{\partial y_{1}} + E_{1212} \frac{\partial \chi_{1}^{11}}{\partial y_{2}} + E_{1221} \frac{\partial \chi_{2}^{11}}{\partial y_{1}} + E_{1122} \frac{\partial \chi_{2}^{11}}{\partial y_{2}} \right) \frac{\partial v_{1}(\mathbf{y})}{\partial y_{1}} \\ + \left( E_{1211} \frac{\partial \chi_{1}^{11}}{\partial y_{1}} + E_{1212} \frac{\partial \chi_{1}^{11}}{\partial y_{2}} + E_{1221} \frac{\partial \chi_{2}^{11}}{\partial y_{1}} + E_{1222} \frac{\partial \chi_{2}^{11}}{\partial y_{2}} \right) \frac{\partial v_{1}(\mathbf{y})}{\partial y_{2}} \\ + \left( E_{2211} \frac{\partial \chi_{1}^{11}}{\partial y_{1}} + E_{2112} \frac{\partial \chi_{1}^{11}}{\partial y_{2}} + E_{2221} \frac{\partial \chi_{2}^{11}}{\partial y_{1}} + E_{2222} \frac{\partial \chi_{2}^{11}}{\partial y_{2}} \right) \frac{\partial v_{2}(\mathbf{y})}{\partial y_{1}} \\ + \left( E_{2211} \frac{\partial \chi_{1}^{11}}{\partial y_{1}} + E_{2212} \frac{\partial \chi_{1}^{11}}{\partial y_{2}} + E_{2221} \frac{\partial \chi_{2}^{11}}{\partial y_{1}} + E_{2222} \frac{\partial \chi_{2}^{11}}{\partial y_{2}} \right) \frac{\partial v_{2}(\mathbf{y})}{\partial y_{2}} \right) \\ = \int_{\pi} \left( E_{1111} \frac{\partial v_{1}(\mathbf{y})}{\partial y_{1}} + E_{1211} \frac{\partial v_{1}(\mathbf{y})}{\partial y_{2}} \right) dY$$
(2.34)

After the simplify the equation (2.34),

$$\int_{\pi} \left\{ \left( E_{1111} \frac{\partial \chi_{1}^{11}}{\partial y_{1}} + E_{1122} \frac{\partial \chi_{2}^{11}}{\partial y_{2}} \right) \frac{\partial v_{1}(\mathbf{y})}{\partial y_{1}} + \left( E_{2211} \frac{\partial \chi_{1}^{11}}{\partial y_{1}} + E_{2222} \frac{\partial \chi_{2}^{11}}{\partial y_{2}} \right) \frac{\partial v_{2}(\mathbf{y})}{\partial y_{2}} \right\} dY$$
$$+ E_{1212} \left( \frac{\partial \chi_{1}^{11}}{\partial y_{2}} + \frac{\partial \chi_{2}^{11}}{\partial y_{1}} \right) \left( \frac{\partial v_{1}(\mathbf{y})}{\partial y_{2}} + \frac{\partial v_{2}(\mathbf{y})}{\partial y_{1}} \right)$$

$$= \int_{\pi} \left( E_{1111} \frac{\partial v_1(\mathbf{y})}{\partial y_1} + E_{2211} \frac{\partial v_2(\mathbf{y})}{\partial y_2} \right) dY$$
(2.35)

Define the cell solution as follows:

$$\chi_i^{11} = \Phi_i, \ i = 1, 2 \tag{2.36}$$

Define the components of the material stiffness matrix as follows:

$$D_{11} = E_{1111} \tag{2.37}$$

$$D_{12} = D_{21} = E_{1122} = E_{2211} \tag{2.38}$$

$$D_{22} = E_{2222} \tag{2.39}$$

$$D_{33} = E_{1212} = E_{1221} = E_{2112} = E_{2121}$$
(2.40)

Then, the material stiffness matrix is shown:

$$\mathbf{D} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{33} \end{bmatrix}$$
(2.41)

And

$$\{d_1\} = \begin{cases} D_{11} \\ D_{21} \\ 0 \end{cases}, \ \{d_2\} = \begin{cases} D_{12} \\ D_{22} \\ 0 \end{cases} \text{ and } \{d_3\} = \begin{cases} 0 \\ 0 \\ D_{33} \end{cases}$$
(2.42)

19

The equation (2.35) can be written as below:

$$\int_{\pi} \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial y_1} & 0 & \frac{\partial}{\partial y_2} \\ 0 & \frac{\partial}{\partial y_2} & \frac{\partial}{\partial y_1} \end{bmatrix} \begin{bmatrix} D \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial y_1} & 0 \\ 0 & \frac{\partial}{\partial y_2} \\ \frac{\partial}{\partial y_2} & \frac{\partial}{\partial y_1} \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix}$$

$$= \int_{\pi} \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial y_1} & 0 & \frac{\partial}{\partial y_2} \\ 0 & \frac{\partial}{\partial y_2} & \frac{\partial}{\partial y_1} \end{bmatrix} \begin{bmatrix} D_{11} \\ D_{12} \\ 0 \end{bmatrix} dY$$

$$(2.43)$$

Further more the arbitrary function  $\mathbf{v}$  and cell solution  $\Phi$  can be expressed with its nodal shape function:

$$\mathbf{v} = \begin{cases} v_1 \\ v_2 \end{cases} = \left[ \beta(x)^N \right]_{2X18} \left\{ v_i^N \right\}_{18X1}, \quad N = 1, 9$$
(2.44)

$$\Phi = \begin{cases} \Phi_1 \\ \Phi_2 \end{cases} = \left[ \beta(x)^N \right]_{2X18} \left\{ \Phi_i^N \right\}_{18X1}, \qquad N = 1, 9 \tag{2.45}$$

$$\mathbf{B} = \begin{bmatrix} \frac{\partial}{\partial y_1} & 0\\ 0 & \frac{\partial}{\partial y_1}\\ \frac{\partial}{\partial y_2} & \frac{\partial}{\partial y_1} \end{bmatrix} \begin{bmatrix} \beta(x)^N \end{bmatrix}_{2x_{18}} = \begin{bmatrix} L \end{bmatrix}_{3x_2} \begin{bmatrix} \beta(x)^N \end{bmatrix}_{2x_{18}}, \quad N=1,9 \quad (2.46)$$

Then, equation (2.43) can be formed as shown below:

$$\int_{\pi} \mathbf{v}^{T} [B]^{T} [D] [B] \{\Phi\} dY = \int_{\pi} \mathbf{v}^{T} [B]^{T} \begin{cases} D_{11} \\ D_{12} \\ 0 \end{cases} dY$$
(2.47)

$$\int_{\pi} [B]^{T} [D] [B] \{\Phi\} dY = \int_{\pi} [B]^{T} \{d_{1}\} dY$$
(2.48)

or

$$[K]{\Phi} = [f] \text{ for the cell solution}$$
(2.49)

where

$$\begin{bmatrix} K \end{bmatrix} = \begin{bmatrix} B \end{bmatrix}^T \begin{bmatrix} D \end{bmatrix} \begin{bmatrix} B \end{bmatrix},$$
$$\begin{bmatrix} f \end{bmatrix} = \int_{\pi} \begin{bmatrix} B \end{bmatrix}^T \{d_1\} dY$$

After obtained the cell solution from the equation (2.49), the homogenized elastic constant defines by using equation (2.33).

$$E_{1111}^{H} = \frac{1}{|Y|} \int_{\pi} \left( E_{1111} - E_{1111} \frac{\partial \chi_{1}^{11}}{\partial y_{1}} - E_{112} \frac{\partial \chi_{1}^{11}}{\partial y_{2}} - E_{1121} \frac{\partial \chi_{2}^{11}}{\partial y_{1}} - E_{1122} \frac{\partial \chi_{2}^{11}}{\partial y_{2}} \right) dY \quad (2.50)$$

where

$$E_{1211}^{H} = 0$$
  
 $E_{2111}^{H} = 0$
Then equation (2.50)  $E_{1111}^{H}$  becomes as shown below:

$$D_{11} = E_{1111}^{H} = \frac{1}{|Y|} \int_{\pi} \left( E_{1111} - E_{1111} \frac{\partial \chi_{1}^{11}}{\partial y_{1}} - E_{1122} \frac{\partial \chi_{2}^{11}}{\partial y_{2}} \right) dY$$

$$= \frac{1}{|Y|} \int_{\pi} \left( D_{11} - D_{11} \frac{\partial \Phi_{1}}{\partial y_{1}} - D_{12} \frac{\partial \Phi_{2}}{\partial y_{2}} \right) dY$$

$$= \frac{1}{|Y|} \int_{\pi} \left( D_{11} - \left[ D_{11} \quad D_{12} \quad 0 \right] \left\{ \frac{\frac{\partial \Phi_{1}}{\partial y_{1}}}{\frac{\partial \Phi_{2}}{\partial y_{2}}} \right\} \right) dY \qquad (2.51)$$

Then, equation (2.51) becomes:

$$D_{11} = \frac{1}{|Y|} \int_{\pi} \left( D_{11} - [d_1]^{\mathrm{T}} \gamma(\Phi) \right) dY$$
(2.52)

where

$$\gamma(\Phi) = \begin{cases} \frac{\partial \Phi_1}{\partial y_1} \\ \frac{\partial \Phi_2}{\partial y_2} \\ \frac{\partial \Phi_1}{\partial y_2} + \frac{\partial \Phi_2}{\partial y_1} \end{cases}$$

For the case of  $E_{2211}$  from equation (2.33) becomes:

$$D_{21}^{H} = E_{2211} = \frac{1}{|Y|} \int_{\pi} \left( E_{2211} - E_{2211} \frac{\partial \chi_{1}^{11}}{\partial y_{1}} - E_{2222} \frac{\partial \chi_{2}^{11}}{\partial y_{2}} \right) dY$$

$$=\frac{1}{|Y|} \int_{\pi} \left( D_{21} - D_{21} \frac{\partial \Phi_1}{\partial y_1} - D_{22} \frac{\partial \Phi_2}{\partial y_2} \right) dY$$
(2.53)

or

$$=\frac{1}{|Y|} \int_{\pi} \left( D_{12} - D_{12} \frac{\partial \Phi_1}{\partial y_1} - D_{22} \frac{\partial \Phi_2}{\partial y_2} \right) dY$$
(2.54)

$$= \frac{1}{|Y|} \int_{\pi} \left( D_{12} - \left[ d_2 \right]^{\mathrm{T}} \left[ \gamma(\Phi) \right] \right) dY$$
(2.55)

Therefore, from the equations (2.52) and (2.55),

$$D_{11}^{H} = \frac{1}{|Y|} \int_{\pi} \left( D_{11} - [d_1]^{\mathrm{T}} \gamma(\Phi) \right) dY$$
 (2.56)

$$D_{21}^{H} = \frac{1}{|Y|} \int_{\pi} \left( D_{12} - [d_{2}]^{\mathrm{T}} \gamma(\Phi) \right) dY$$
(2.57)

where

$$\gamma(\Phi) = \begin{cases} \frac{\partial \Phi_1}{\partial y_1} \\ \frac{\partial \Phi_2}{\partial y_2} \\ \frac{\partial \Phi_1}{\partial y_2} + \frac{\partial \Phi_2}{\partial y_1} \end{cases}$$

Equations (2.49), (2.56) and (2.57) will be used for developing the finite element homogenization algorithm.

<u>Case B: k = l = 2</u>

Expand the equation (2.32) and removing terms with zero coefficients,

$$\int_{\pi} \left\{ \begin{array}{l} \left( E_{1111} \frac{\partial \chi_{1}^{22}}{\partial y_{1}} + E_{1122} \frac{\partial \chi_{2}^{22}}{\partial y_{2}} \right) \frac{\partial v_{1}(\mathbf{y})}{\partial y_{1}} + \left( E_{2211} \frac{\partial \chi_{1}^{22}}{\partial y_{1}} + E_{2222} \frac{\partial \chi_{2}^{22}}{\partial y_{2}} \right) \frac{\partial v_{2}(\mathbf{y})}{\partial y_{2}} \\ + E_{1212} \left( \frac{\partial \chi_{1}^{22}}{\partial y_{2}} + E_{1221} \frac{\partial \chi_{2}^{22}}{\partial y_{1}} \right) \left( \frac{\partial v_{1}(\mathbf{y})}{\partial y_{2}} + \frac{\partial v_{2}(\mathbf{y})}{\partial y_{1}} \right) \end{array} \right\} dY$$

$$= \int_{\pi} \left( E_{1122} \frac{\partial v_1(\mathbf{y})}{\partial y_2} + E_{2222} \frac{\partial v_2(\mathbf{y})}{\partial y_2} \right) dY$$
(2.58)

Define the cell solution as follows:

$$\chi_i^{22} = \Psi_i, \ i = 1, 2 \tag{2.59}$$

Then, equation (2.58) can be form as below:

$$\int_{\pi} [B]^{T} [D] [B] \{\Psi\} dY = \int_{\pi} [B]^{T} \{d_{2}\} dY$$
(2.60)

or

$$[K]{\Psi} = [f]$$
 for the cell solution (2.61)

where

$$[K] = [B]^T [D][B],$$
$$[f] = \int_{\pi} [B]^T \{d_2\} dY$$

After obtained the cell solution from the equation (2.61), the homogenized elastic constant defines by using equation (2.33).

$$E_{1122}^{H} = \frac{1}{|Y|} \int_{\pi} \left( E_{1122} - E_{1111} \frac{\partial \chi_{1}^{22}}{\partial y_{1}} - E_{1122} \frac{\partial \chi_{2}^{22}}{\partial y_{2}} \right) dY$$
(2.62)

where

$$E_{1222}^{H} = 0$$

$$E_{2122}^{H} = 0$$

$$E_{2222}^{H} = \frac{1}{|Y|} \int_{\pi} \left( E_{2222} - E_{2211} \frac{\partial \chi_{1}^{22}}{\partial y_{1}} - E_{2222} \frac{\partial \chi_{2}^{22}}{\partial y_{2}} \right) dY \qquad (2.63)$$

Apply same process as it shown from previous case. Then equations (2.62) and (2.63) become:

$$D_{12}^{H} = E_{1122}^{H} = \frac{1}{|Y|} \int_{\pi} \left( D_{12} - [d_1]^{\mathrm{T}} \gamma(\Psi) \right) dY$$
(2.64)

$$D_{22}^{H} = E_{2222}^{H} = \frac{1}{|Y|} \int_{\pi} \left( D_{22} - [d_{2}]^{\mathrm{T}} \gamma(\Psi) \right) dY$$
(2.65)

where

$$\gamma(\Psi) = \begin{cases} \frac{\partial \Psi_1}{\partial y_1} \\ \frac{\partial \Psi_2}{\partial y_2} \\ \frac{\partial \Psi_1}{\partial y_2} + \frac{\partial \Psi_2}{\partial y_1} \end{cases}$$

Equations (2.61), (2.64) and (2.65) will be used for developing the finite element homogenization algorithm.

<u>Case C: k = 1, l = 2 (or k = 2, l = 1)</u>

$$\int_{\pi} \left\{ \begin{pmatrix} E_{1111} \frac{\partial \chi_{1}^{12}}{\partial y_{1}} + E_{1122} \frac{\partial \chi_{2}^{12}}{\partial y_{2}} \end{pmatrix} \frac{\partial v_{1}(\mathbf{y})}{\partial y_{1}} + \begin{pmatrix} E_{2211} \frac{\partial \chi_{1}^{12}}{\partial y_{1}} + E_{2222} \frac{\partial \chi_{2}^{12}}{\partial y_{2}} \end{pmatrix} \frac{\partial v_{2}(\mathbf{y})}{\partial y_{2}} \\ + E_{1212} \left( \frac{\partial \chi_{1}^{12}}{\partial y_{2}} + E_{1221} \frac{\partial \chi_{2}^{12}}{\partial y_{1}} \right) \left( \frac{\partial v_{1}(\mathbf{y})}{\partial y_{2}} + \frac{\partial v_{2}(\mathbf{y})}{\partial y_{1}} \right) \\ = \int_{\pi} \left( E_{1212} \left( \frac{\partial v_{1}(\mathbf{y})}{\partial y_{2}} + \frac{\partial v_{2}(\mathbf{y})}{\partial y_{1}} \right) \right) dY$$
(2.66)

Define the cell solutions as follows:

$$\chi_i^{12} = \Theta_i, \ i = 1, 2 \tag{2.67}$$

Then, equation (2.66) can be form as below:

$$\int_{\pi} [B]^{T} [D] [B] \{\Theta\} dY = \int_{\pi} [B]^{T} \{d_{3}\} dY$$
(2.68)

or

$$[K]{\Theta} = [f] \text{ for the cell solution}$$
(2.69)

where

$$\begin{bmatrix} K \end{bmatrix} = \begin{bmatrix} B \end{bmatrix}^T \begin{bmatrix} D \end{bmatrix} \begin{bmatrix} B \end{bmatrix},$$
$$\begin{bmatrix} f \end{bmatrix} = \int_{\pi} \begin{bmatrix} B \end{bmatrix}^T \{d_3\} dY$$

After obtained the cell solution from the equation (2.69), the homogenized elastic constant defines by using equation (2.33).

$$E_{1112}^{H} = E_{1121}^{H} = 0$$

$$E_{1212}^{H} = E_{1221}^{H} = \frac{1}{|Y|} \int_{\pi} \left( E_{1212} - E_{1212} \frac{\partial \chi_{1}^{12}}{\partial y_{2}} - E_{1221} \frac{\partial \chi_{2}^{12}}{\partial y_{1}} \right) dY \qquad (2.70)$$

$$E_{2112}^{H} = E_{2121}^{H} = \frac{1}{|Y|} \int_{\pi} \left( E_{2112} - E_{2112} \frac{\partial \chi_{1}^{12}}{\partial y_{2}} - E_{2121} \frac{\partial \chi_{2}^{12}}{\partial y_{1}} \right) dY$$
(2.71)  
$$E_{2212}^{H} = E_{2221}^{H} = 0$$

From the results of equations (2.70) and (2.71), it shows that

$$E_{1212}^{H} = E_{1221}^{H} = E_{2112}^{H} = E_{2121}^{H}$$
(2.72)

Then, equations (2.70) and (2.71) become,

$$D_{33}^{H} = \frac{1}{|Y|} \int_{\pi} \left( D_{33} - [d_{3}]^{\mathrm{T}} \gamma(\Theta) \right) dY$$
(2.73)

where

$$\gamma(\Theta) = \begin{cases} \frac{\partial \Theta_1}{\partial y_1} \\ \frac{\partial \Theta_2}{\partial y_2} \\ \frac{\partial \Theta_1}{\partial y_2} + \frac{\partial \Theta_2}{\partial y_1} \end{cases}$$

Equations (2.69) and (2.73) will be used for developing the finite element homogenization algorithm.

Finally, the homogenized material stiffness matrix  $\mathbf{D}^{H}$  is defined and shown below. It will use in conventional finite element analysis process:

$$\begin{bmatrix} \mathbf{D}^{H} \end{bmatrix} = \begin{bmatrix} D_{11}^{H} & D_{12}^{H} & 0 \\ D_{12}^{H} & D_{22}^{H} & 0 \\ 0 & 0 & D_{33}^{H} \end{bmatrix}$$
(2.74)

## 2.3 Macro Homogenized Solution

The conventional stiffness matrix for 2-D plane stress element is shown below:

$$[k] = \iint [\mathbf{B}]^T [\mathbf{E}] [\mathbf{B}] t dx dy$$
(2.75)

where

$$\begin{bmatrix} \mathbf{B} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial y_1} & 0\\ 0 & \frac{\partial}{\partial y_1} \\ \frac{\partial}{\partial y_2} & \frac{\partial}{\partial y_1} \end{bmatrix} \begin{bmatrix} \beta(x)^N \end{bmatrix}_{2x_{18}} = \begin{bmatrix} L \end{bmatrix}_{3x_2} \begin{bmatrix} \beta(x)^N \end{bmatrix}_{2x_{18}}, \qquad N = 1, 9$$

[E] = Material stiffness matrix

t = Material thickness

Convert the equation (2.75) from Cartesian coordinate system dxdy to

Isoparametric coordinate systems  $d\xi d\eta$ . The sides of the element limitation are at  $\xi = \pm 1$  and at  $\eta = \pm 1$ .

$$[k] = \iint [\mathbf{B}]^T [\mathbf{E}] [\mathbf{B}] t dx dy = \iint_{-1-1}^{1} [\mathbf{B}]^T [\mathbf{E}] [\mathbf{B}] t \mathbf{J} d\xi d\eta$$
(2.76)

where **J** is called the Jacobian matrix which is defined as below:

$$\mathbf{J} = \begin{bmatrix} \sum \boldsymbol{\beta}_{i,\xi}^{N} x_{i} & \sum \boldsymbol{\beta}_{i,\xi}^{N} y_{i} \\ \sum \boldsymbol{\beta}_{i,\eta}^{N} x_{i} & \sum \boldsymbol{\beta}_{i,\eta}^{N} y_{i} \end{bmatrix}$$
(2.77)

Finally, replace the stiffness matrix  $[\mathbf{E}]$  with  $[\mathbf{D}^{H}]$ . Then the equation (2.76) becomes:

$$\begin{bmatrix} k \end{bmatrix}^{H} = \int_{-1-1}^{1} \begin{bmatrix} \mathbf{B} \end{bmatrix}^{T} \begin{bmatrix} \mathbf{D}^{H} \end{bmatrix} \begin{bmatrix} \mathbf{B} \end{bmatrix} t \, \mathbf{J} d\xi d\eta$$
(2.78)

## 2.4 Variation of Cell Solution $\chi_i^{kl}$ in Microcell Geometries

2.4.1 A Typical Shape of Microcell Geometry

Based on the equations (2.32) and (2.33), homogenization program was created and tested. Two cell geometries, as pictured in Figure 2.4 were defined. The microcell structures have soft and hard materials,  $E_{soft}$  and  $E_{hard}$ . In the test cases, it was assumed that  $E_{soft} = 10$ ,  $E_{hard} = 1000$  and Poisson ratio v = 0.3. The shaded area represents  $E_{hard}$  which is a fiber and the brightened area represents  $E_{soft}$ which is a matrix. The microcell models are pictured in Figure 2.4. The fiber width  $hx_1$  and  $hy_1 = 0.250$  for both Model 1 and 2. The matrix lengths in Model 1 were  $hx_2 = hy_2 = 0.375$  and Model 2 were  $hx_2 = 0.125$  and  $hy_2 = 0.375$ . The cell solution  $\chi_i^{kl}$  was computed for the two cell geometry cases.



Figure 2.4 Microcell Structure: Model 1 - Cell Size 1.0 x 1.0; Model 2 - Cell size 0.5 x 1.0

2.4.2 Computation of the Cell Solution ( $\chi_i^{kl}$ ) in Soft and Hard Microcell

Structure Cases

Case A: k=l=1

Figure 2.5 and Figure 2.6 showed the results of the microcell solution for Model 1 and Model 2 after the homogenization program was ran. Model 1 has the cell size as  $1.0 \times 1.0$  and Model 2 has  $0.5 \times 1.0$ .



Figure 2.5 Microcell Solution for Model 1 – Cell size 1.0 x 1.0



Figure 2.6 Microcell Solution for Model 2 – Cell size 0.5 x 1.0

The pictures of Figure 2.5 and Figure 2.6 were superimposed in order to demonstrate that the microcell solutions are different if the microstructural shapes are different (see Figure 2.7).



Figure 2.7 Superimpose the Cell Solution Pictures of Figure 2.5 and Figure 2.6

<u>Case B: k = l = 2</u>

Figure 2.8 and Figure 2.9 showed the results of the microcell solution for Model 1 and Model 2 after the homogenization program was ran.



Figure 2.8 Microcell Solution for Model 1 – Cell size 1.0 x 1.0



Figure 2.9 Microcell Solution for Model 2 - Cell size 0.5 x 1.0

Again, the pictures of Figure 2.8 and Figure 2.9 were superimposed in order to demonstrate that the microcell solutions are different if the microstructural shapes are different (see Figure 2.10)



Figure 2.10 Superimpose the Cell Solution Pictures of Figure 2.8 and Figure 2.9

<u>Case C: k = 1, l = 2 (or k = 2, l = 1)</u>

Figure 2.11 and Figure 2.12 showed the results of the microcell solution for

Model 1 and Model 2 after the homogenization program was ran.



Figure 2.11 Microcell Solution for Model 1 – Cell Size 1.0 x 1.0



Figure 2.12 Microcell Solution for Model 2 – Cell Size 0.5 x 1.0

Finally, the pictures of Figure 2.11 and Figure 2.12 were superimposed in order to demonstrate that the microcell solutions are different if the microstructural shapes are different (see Figure 2.13).



Figure 2.13 Superimpose the Cell Solution Pictures of Figure 2.11 and Figure 2.12

### 2.5 Verification of the Homogenized Elastic Constants

#### 2.5.1 Soft and Hard Isotropic Composite Material with Variation

A finite element cell solution program HOMOG.f90, was created to obtain approximate cell solutions. The homogenized elastic constant,  $E_{ijkl}^{H}$ , was computed and compared with the results from Bendoe and Kikuchi [3]. For the first case, the cell had the soft and hard material properties with isotropic and plane stress. And the soft material's Young's modulus  $E_{soft} = 10$  and  $E_{hard} = 1000$  with the same Poisson ratio v = 0.3. The structure is pictured in Figure 2.14.

E <sub>soft</sub>	E <sub>hard</sub>	E <sub>soft</sub>
E <sub>hard</sub>	E <sub>soft</sub>	E <sub>hard</sub>
E <sub>soft</sub>	E <sub>hard</sub>	E <sub>soft</sub>

Figure 2.14 Material Properties in a Single Microcell Geometry:  $(E_{soft} = 10 \text{ and } E_{hard} = 1000)$ 

Bendoe and Kikuchi [3] calculated the homogenized elastic constant with a 16x16 mesh of elements on the cell picture in Figure 2.15. They also used an adaptive method to obtain refined the results.

HOMOG.f90 used an 8 x 8 mesh of elements of the cell geometry which is shown in Figure 2.16 for calculating the homogenized elastic constant. A ninenode Lagrange quadratic element and sixteen Gauss points were used to increase accuracy. The comparison results are shown in Table 2.1.



Figure 2.15 A Single Microcell Geometry Model Defined by Bendoe and Kikuchi [3]: 16 x 16 Uniform Square Four-node Isoparametric Elements



Figure 2.16 A Single Microcell Geometry Model Defined by HOMOG.f90: 8 x 8 Uniform Square Nine-node Lagrange Quadratic Elements

Mesh or Nodes	$E_{1111}^{H}$	$E_{_{1122}}^{_{H}}$	$E_{_{22222}}^{_{H}}$	$E_{1212}^{H}$
16 x 16 <sup>[3]</sup>	149.80	71.61	149.80	87.12
1 <sup>st</sup> Adapt <sup>[3]</sup>	127.12	62.91	127.12	75.90
2 <sup>nd</sup> Adapt <sup>[3]</sup>	125.79	62.62	125.79	75.28
HOMOG.f90 (9-node)	136.55	68.56	136.55	81.07

Table 2.1 Comparison of the Homogenized Elastic Constants: Soft and Hard Material

2.5.2 A Rectangular Hole in the Material

For the second case verification, Bendoe and Kikuchi [3] used rectangular microcell geometries with a rectangular hole in the middle. The structure is pictured in Figure 2.17 and has material properties which are characterized by  $E_{1111} = E_{2222} = 30$  and  $E_{1122} = E_{1212} = 10$ . In this case, the hole was not approximated by a very soft material. The material properties of the microcell only consider the solid area.



Figure 2.17 Material Properties in a Single Microcell Geometry with a Hole:  $(E_{1111} = E_{2222} = 30 \text{ and } E_{1122} = E_{1212} = 10)$ 

In Figure 2.18, a 20 x 20 mesh size was used for the cell geometry and three different adaptation mesh methods were generated to obtain refined the results of homogenized elastic constant.

HOMOG.f90 used a  $10 \times 10$  mesh of elements of the cell geometry which is shown in Figure 2.19 for calculating the homogenized elastic constant. As same as the first case, a nine-node Lagrange quadratic element and sixteen Gauss points were used to increase accuracy. The comparison results are shown in Table 2.2.



Figure 2.18 A Single Microcell Geometry Model Defined by Bendoe and Kikuchi [3]: 20 x 20 Uniform Square Four-node Isoparametric Element



Figure 2.19 A Single Microcell Geometry Model Defined by HOMOG.f90: 10 x 10 Uniform Square Nine-node Lagrange Quadratic Elements

Mesh or Nodes	$E_{1111}^{H}$	$E_{1122}^{H}$	$E_{2222}^{H}$	$E_{1212}^{H}$
16 x 16 <sup>[3]</sup>	13.015	3.241	17.552	2.785
1 <sup>st</sup> Adapt <sup>[3]</sup>	12.910	3.178	17.473	2.714
2 <sup>nd</sup> Adapt <sup>[3]</sup>	12.865	3.146	17.437	2.683
3 <sup>rd</sup> Adapt <sup>[3]</sup>	12.844	3.131	17.421	2.668
HOMOG.f90 (9-node)	12.924	3.198	17.487	2.708

Table 2.2 Elastic Tensors after Homogenization in Rectangular Hole

# Chapter 3 Nonperiodic Homogenization (NPH) Method

## 3.1 Mathematical Theory of NPH

Assume that the microscopic and the macroscopic have a relationship based on equations (2.24) and (2.30). Then an equation can be written as below:

$$u_{i}(\mathbf{x}, \mathbf{y}) = u_{i}(\mathbf{x}) + \chi_{(i)}^{kl}(\mathbf{x}, \mathbf{y}) \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{l}} \overline{C}_{i}(\mathbf{x})$$
Macro Structure Micro Structure
Solution Solution
(3.1)

where,  $\chi_i^{kl}(\mathbf{x}, \mathbf{y}) =$  Microcell solution (*i* = 1, 2)

 $\overline{C}_i(\mathbf{x})$  = Correction coefficient related to the micro structure

 $u_k^H(\mathbf{x})$  = Homogenized displacement solution (k = 1, 2)

 $X_i$  = Macro coordinate system (i = 1, 2)

Also,  $u_i(\mathbf{x})$  can be expressed as function of the homogenized displacement with

correction coefficient  $C_i(\mathbf{x})$  and  $\overline{\overline{C}}_i(\mathbf{x})$  through the expression below:

$$u_i(\mathbf{x}) = u_{(i)}^H(\mathbf{x})C_i(\mathbf{x}) + \overline{C}_i(\mathbf{x})$$
(3.2)

Then, insert in the equation (3.2) into equation (3.1) then can be rewritten as below:

$$u_{i}(\mathbf{x},\mathbf{y}) = u_{(i)}^{H}(\mathbf{x})C_{i}(\mathbf{x}) + \overline{C}_{i}(\mathbf{x}) + \chi_{(i)}^{kl}(\mathbf{x},\mathbf{y})\frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{l}}\overline{C}_{i}(\mathbf{x})$$
(3.3)

Now, the nodeless form of the equation can be derived. Let at  $\mathbf{x}_N$  be a nodal position in the macro element. Let  $\mathbf{y}_c$  be a single location in the microcell. Evaluating equation (3.3) at  $\mathbf{x}_N$  and  $\mathbf{y}_c$  produces the follow equations:

$$u_{i}(\mathbf{x},\mathbf{y})|_{\substack{\mathbf{x}=\mathbf{x}_{N}\\\mathbf{y}=\mathbf{y}_{C}}} = u_{(i)}^{H}(\mathbf{x}_{N})C_{i}(\mathbf{x}_{N}) + \overline{C}_{i}(\mathbf{x}_{N}) + \chi_{(i)}^{kl}(\mathbf{x}_{N},\mathbf{y}_{C})\frac{\partial u_{k}^{H}(\mathbf{x}_{N})}{\partial X_{l}}\overline{C}_{i}(\mathbf{x}_{N}) \quad (3.4)$$

Therefore, displacement components  $u_i$  at nod N can be formed as shown below:

$$u_i^N = u_{(i)}^H(\mathbf{x}_N)C_i^N + \overline{\overline{C}}_i^N + \chi_{(i)}^{kl}(\mathbf{x}_N, \mathbf{y}_C) \frac{\partial u_k^H(\mathbf{x}_N)}{\partial X_l} \overline{C}_i^N$$
(3.5)

Then, correction coefficient component  $\overline{C}_i^N$ , can be defined as a function of the displacement associates with the cell solution as follows:

$$\overline{\overline{C}}_{i}^{N} = u_{i}^{N} - u_{(i)}^{H}(\mathbf{x}_{N})C_{i}^{N} - \chi_{(i)}^{kl}(\mathbf{x}_{N},\mathbf{y}_{C})\frac{\partial u_{k}^{H}(\mathbf{x}_{N})}{\partial X_{l}}\overline{C}_{i}^{N}$$
(3.6)

Now introduce equation (3.6) into the equation (3.4) and rearrange the resulting expression in terms of the correction coefficients  $C_i(\mathbf{x})$  and  $\overline{C}_i(\mathbf{x})$ .

$$u_{i}(\mathbf{x}, \mathbf{y}) = u_{(i)}^{H}(\mathbf{x})C_{i}(\mathbf{x}) + \chi_{(i)}^{kl}(\mathbf{x}, \mathbf{y})\frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{l}}\overline{C}_{i}(\mathbf{x})$$
$$+ \beta^{N}(\mathbf{x})[u_{i}^{N} - u_{(i)}^{H}(\mathbf{x}_{N})C_{i}^{N} - \chi_{(i)}^{kl}(\mathbf{x}_{N}, \mathbf{y}_{C})\frac{\partial u_{k}^{H}(\mathbf{x}_{N})}{\partial X_{l}}\overline{C}_{i}^{N}]$$
(3.7)

where  $C_i(\mathbf{x}) = \beta^N(\mathbf{x})C_i^N$ 

$$\overline{C}_{i}(\mathbf{x}) = \beta^{N}(\mathbf{x})\overline{C}_{i}^{N}$$

$$u_{i}(\mathbf{x}) = \beta^{N}(\mathbf{x})u_{i}^{N}$$
(3.8)

Figure 3.1 indicates the nodal degree-of-freedom for the 1-D element case. Each nodal point has three degree-of-freedom  $C_i, \overline{C}_i$  and  $u_i$ .



Figure 3.1 Nodal Degree of Freedom in 1-D Element

Introduce equation (3.8) into equation (3.7) to obtain

$$u_{i}(\mathbf{x}, \mathbf{y}) = u_{(i)}^{H}(\mathbf{x})\beta^{N}(\mathbf{x})C_{i}^{N} + \chi_{(i)}^{kl}(\mathbf{x}, \mathbf{y})\frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{l}}\beta^{N}(\mathbf{x})\overline{C}_{i}^{N}$$
$$+ \beta^{N}(\mathbf{x})\left[u_{i}^{N} - u_{(i)}^{H}(\mathbf{x}_{N})C_{i}^{N} - \chi_{(i)}^{kl}(\mathbf{x}_{N}, \mathbf{y}_{C})\frac{\partial u_{k}^{H}(\mathbf{x}_{N})}{\partial X_{l}}\overline{C}_{i}^{N}\right]$$
(3.9)

Finally, the displacement vector  $u_i(\mathbf{x}, \mathbf{y})$  can be formed as shown below:

$$u_{i}(\mathbf{x}, \mathbf{y}) = \left\{ u_{(i)}^{H}(\mathbf{x}) \beta^{N}(\mathbf{x}) - u_{(i)}^{H}(\mathbf{x}_{N}) \beta^{N}(\mathbf{x}) \right\} C_{i}^{N}$$
$$+ \left\{ \chi_{(i)}^{kl}(\mathbf{x}, \mathbf{y}) \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{l}} \beta^{N}(\mathbf{x}) - \chi_{(i)}^{kl}(\mathbf{x}_{N}, \mathbf{y}_{C}) \frac{\partial u_{k}^{H}(\mathbf{x}_{N})}{\partial X_{l}} \beta^{N}(\mathbf{x}) \right\} \overline{C}_{i}^{N}$$

$$+\beta^{N}(\mathbf{x})u_{i}^{N} \tag{3.10}$$

or

$$u_i(\mathbf{x}, \mathbf{y}) = \gamma^N(\mathbf{x})C_i^N + \alpha^N(\mathbf{x})\overline{C}_i^N + \beta^N(\mathbf{x})u_i^N$$
(3.11)

where,  $\gamma^{N}(\mathbf{x})$  and  $\alpha^{N}(\mathbf{x})$  are nodeless shape functions:

$$\gamma^{N} = u_{(i)}^{H}(\mathbf{x})\beta^{N}(\mathbf{x}) - u_{(i)}^{H}(\mathbf{x}_{N})\beta^{N}(\mathbf{x})$$
$$\alpha^{N} = \chi_{(i)}^{kl}(\mathbf{x}, \mathbf{y})\frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{l}}\beta^{N}(\mathbf{x}) - \chi_{(i)}^{kl}(\mathbf{x}_{N}, \mathbf{y}_{C})\frac{\partial u_{k}^{H}(\mathbf{x}_{N})}{\partial X_{l}}\beta^{N}(\mathbf{x})$$

## 3.2 Evaluation of the NPH Strain Tensors

The two dimensional strains in Cartesian coordinates are defined as follows:

$$e_{11} = \frac{\partial u_1(\mathbf{x}, \mathbf{y})}{\partial X_1} + \frac{\partial u_1(\mathbf{x}, \mathbf{y})}{\partial Y_1} \frac{\partial Y_1}{\partial X_1}$$
(3.12)

$$e_{22} = \frac{\partial u_2(\mathbf{x}, \mathbf{y})}{\partial X_2} + \frac{\partial u_2(\mathbf{x}, \mathbf{y})}{\partial Y_2} \frac{\partial Y_2}{\partial X_2}$$
(3.13)

$$e_{12} = \frac{\partial u_1(\mathbf{x}, \mathbf{y})}{\partial X_2} + \frac{\partial u_1(\mathbf{x}, \mathbf{y})}{\partial Y_2} \frac{\partial Y_2}{\partial X_2} + \frac{\partial u_2(\mathbf{x}, \mathbf{y})}{\partial X_1} + \frac{\partial u_2(\mathbf{x}, \mathbf{y})}{\partial Y_1} \frac{\partial Y_1}{\partial X_1}$$
(3.14)

Equation (3.11) through equation (3.13) can be rearranged into matrix format as follows:

$$\begin{cases} e_{11} \\ e_{22} \\ e_{12} \end{cases} = \begin{cases} \frac{\partial u_1(\mathbf{x}, \mathbf{y})}{\partial X_1} + \frac{\partial u_1(\mathbf{x}, \mathbf{y})}{\partial Y_1} \frac{\partial Y_1}{\partial X_1} \\ \frac{\partial u_2(\mathbf{x}, \mathbf{y})}{\partial X_2} + \frac{\partial u_2(\mathbf{x}, \mathbf{y})}{\partial Y_2} \frac{\partial Y_2}{\partial X_2} \\ \frac{\partial u_1(\mathbf{x}, \mathbf{y})}{\partial X_2} + \frac{\partial u_1(\mathbf{x}, \mathbf{y})}{\partial Y_2} \frac{\partial Y_2}{\partial X_2} + \frac{\partial u_2(\mathbf{x}, \mathbf{y})}{\partial X_1} + \frac{\partial u_2(\mathbf{x}, \mathbf{y})}{\partial Y_1} \frac{\partial Y_1}{\partial X_1} \end{cases} \\ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \mathbf{x} \begin{bmatrix} \frac{\partial u_1(\mathbf{x}, \mathbf{y})}{\partial X_2} \\ \frac{\partial u_1(\mathbf{x}, \mathbf{y})}{\partial X_2} \\ \frac{\partial u_2(\mathbf{x}, \mathbf{y})}{\partial X_2} \\ \frac{\partial u_2(\mathbf{x}, \mathbf{y})}{\partial X_1} \\ \frac{\partial u_2(\mathbf{x}, \mathbf{y})}{\partial X_2} \end{bmatrix} + \begin{bmatrix} \frac{\partial u_1(\mathbf{x}, \mathbf{y})}{\partial Y_1} \frac{1}{\partial x_1} \\ \frac{\partial u_2(\mathbf{x}, \mathbf{y})}{\partial Y_1} \frac{1}{\partial x_1} \\ \frac{\partial u_2(\mathbf{x}, \mathbf{y})}{\partial Y_2} \frac{1}{\partial Y_2} \\ \frac{\partial u_2(\mathbf{x}, \mathbf{y})}{\partial Y_2} \\ \frac{\partial u_2(\mathbf{x}, \mathbf{y})}{\partial Y_2} \frac{1}{\partial Y_2} \\ \frac{\partial u_2(\mathbf{x}, \mathbf{y})}{\partial Y_2} \frac{1}{\partial Y_2} \\ \frac{\partial u_2(\mathbf{x}, \mathbf{y})}{\partial Y_2} \frac{1}{\partial Y_2} \\ \frac{\partial u_2(\mathbf{x}, \mathbf{y})}{\partial Y_2} \\ \frac{\partial u_2(\mathbf{$$

where,

$$\frac{1}{\varepsilon_1} = \frac{\partial Y_1}{\partial X_1} \text{ and } \frac{1}{\varepsilon_2} = \frac{\partial Y_2}{\partial X_2}$$
(3.16)

Now, substitute equation (3.10) into equation (3.15). Some of the terms

will become zero after differentiation. For example,  $\frac{\partial u_1^H(\mathbf{x}_{(N)})}{\partial X_1}$  is a constant

relative to variations in  $X_1$  and  $X_2$ . Set value of cell solution at  $\mathbf{y}_c$  to zero (i.e.  $\chi_1^{kl}(\mathbf{x}_{(N)}, \mathbf{y}_c) = 0$ ). Rearrange the equation with respect to common coefficient terms, then the equation (3.15) becomes as shown below:

$$\begin{cases} e_{11} \\ e_{22} \\ e_{12} \end{cases} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} u_1^H(\mathbf{x}) \frac{\partial \beta^N(\mathbf{x})}{\partial X_1} C_1^N \\ u_1^H(\mathbf{x}) \frac{\partial \beta^N(\mathbf{x})}{\partial X_2} C_1^N \\ u_2^H(\mathbf{x}) \frac{\partial \beta^N(\mathbf{x})}{\partial X_1} C_2^N \\ u_2^H(\mathbf{x}) \frac{\partial \beta^N(\mathbf{x})}{\partial X_2} C_2^N \end{bmatrix} - \begin{bmatrix} u_1^H(\mathbf{x}_{(N)}) \frac{\partial \beta^N(\mathbf{x})}{\partial X_1} C_1^N \\ u_1^H(\mathbf{x}_{(N)}) \frac{\partial \beta^N(\mathbf{x})}{\partial X_2} C_1^N \\ u_2^H(\mathbf{x}_{(N)}) \frac{\partial \beta^N(\mathbf{x})}{\partial X_1} C_2^N \\ u_2^H(\mathbf{x}_{(N)}) \frac{\partial \beta^N(\mathbf{x})}{\partial X_2} C_2^N \end{bmatrix} - \begin{bmatrix} u_1^H(\mathbf{x}_{(N)}) \frac{\partial \beta^N(\mathbf{x})}{\partial X_1} C_1^N \\ u_2^H(\mathbf{x}_{(N)}) \frac{\partial \beta^N(\mathbf{x})}{\partial X_1} C_2^N \\ u_2^H(\mathbf{x}_{(N)}) \frac{\partial \beta^N(\mathbf{x})}{\partial X_2} C_2^N \end{bmatrix}$$

$$+ \begin{cases} \frac{\partial u_{1}^{H}(\mathbf{x})}{\partial X_{1}} \beta^{N}(\mathbf{x})C_{1}^{N} - \frac{\partial u_{1}^{H}(\mathbf{x}_{(N)})}{\partial X_{1}} \beta^{N}(\mathbf{x})C_{1}^{N} \\ \frac{\partial u_{1}^{H}(\mathbf{x})}{\partial X_{2}} \beta^{N}(\mathbf{x})C_{1}^{N} - \frac{\partial u_{1}^{H}(\mathbf{x}_{(N)})}{\partial X_{2}} \beta^{N}(\mathbf{x})C_{1}^{N} \\ \frac{\partial u_{2}^{H}(\mathbf{x})}{\partial X_{1}} \beta^{N}(\mathbf{x})C_{2}^{N} - \frac{\partial u_{2}^{H}(\mathbf{x}_{(N)})}{\partial X_{1}} \beta^{N}(\mathbf{x})C_{2}^{N} \\ \frac{\partial u_{2}^{H}(\mathbf{x})}{\partial X_{2}} \beta^{N}(\mathbf{x})C_{2}^{N} - \frac{\partial u_{2}^{H}(\mathbf{x}_{(N)})}{\partial X_{2}} \beta^{N}(\mathbf{x})C_{2}^{N} \end{cases}$$

$$+ \begin{cases} \chi_{1}^{kl}(\mathbf{x},\mathbf{y}) \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{1}} \frac{\partial \beta^{N}(\mathbf{x})}{\partial X_{1}} \overline{C}_{1}^{N} - \chi_{1}^{kl}(\mathbf{x}_{(N)},\mathbf{y}_{c}) \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{1}} \frac{\partial \beta^{N}(\mathbf{x})}{\partial X_{1}} \overline{C}_{1}^{N} \\ \chi_{1}^{kl}(\mathbf{x},\mathbf{y}) \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{1}} \frac{\partial \beta^{N}(\mathbf{x})}{\partial X_{2}} \overline{C}_{1}^{N} - \chi_{1}^{kl}(\mathbf{x}_{(N)},\mathbf{y}_{c}) \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{1}} \frac{\partial \beta^{N}(\mathbf{x})}{\partial X_{2}} \overline{C}_{1}^{N} \\ \chi_{2}^{kl}(\mathbf{x},\mathbf{y}) \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{1}} \frac{\partial \beta^{N}(\mathbf{x})}{\partial X_{1}} \overline{C}_{2}^{N} - \chi_{2}^{kl}(\mathbf{x}_{(N)},\mathbf{y}_{c}) \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{1}} \frac{\partial \beta^{N}(\mathbf{x})}{\partial X_{1}} \overline{C}_{2}^{N} \\ \chi_{2}^{kl}(\mathbf{x},\mathbf{y}) \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{1}} \frac{\partial \beta^{N}(\mathbf{x})}{\partial X_{2}} \overline{C}_{2}^{N} - \chi_{2}^{kl}(\mathbf{x}_{(N)},\mathbf{y}_{c}) \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{1}} \frac{\partial \beta^{N}(\mathbf{x})}{\partial X_{2}} \overline{C}_{2}^{N} \\ \end{cases}$$

$$+ \begin{cases} \frac{\partial \chi_{1}^{kl}(\mathbf{x},\mathbf{y})}{\partial X_{1}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{l}} \beta^{N}(\mathbf{x}) \overline{C}_{1}^{N} \\ \frac{\partial \chi_{1}^{kl}(\mathbf{x},\mathbf{y})}{\partial X_{2}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{l}} \beta^{N}(\mathbf{x}) \overline{C}_{1}^{N} \\ \frac{\partial \chi_{2}^{kl}(\mathbf{x},\mathbf{y})}{\partial X_{2}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{l}} \beta^{N}(\mathbf{x}) \overline{C}_{2}^{N} \\ \frac{\partial \chi_{2}^{kl}(\mathbf{x},\mathbf{y})}{\partial X_{2}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{l}} \beta^{N}(\mathbf{x}) \overline{C}_{2}^{N} \\ \frac{\partial \chi_{2}^{kl}(\mathbf{x},\mathbf{y})}{\partial X_{2}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{l}} \beta^{N}(\mathbf{x}) \overline{C}_{2}^{N} \\ \frac{\partial \chi_{2}^{kl}(\mathbf{x},\mathbf{y})}{\partial X_{2}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{l}} \beta^{N}(\mathbf{x}) \overline{C}_{2}^{N} \\ \end{cases} + \begin{cases} \chi_{1}^{kl}(\mathbf{x},\mathbf{y}) \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{2} \partial X_{l}} \beta^{N}(\mathbf{x}) \overline{C}_{1}^{N} \\ \chi_{2}^{kl}(\mathbf{x},\mathbf{y}) \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{1} \partial X_{l}} \beta^{N}(\mathbf{x}) \overline{C}_{2}^{N} \\ \chi_{2}^{kl}(\mathbf{x},\mathbf{y}) \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{2} \partial X_{l}} \beta^{N}(\mathbf{x}) \overline{C}_{2}^{N} \\ \chi_{2}^{kl}(\mathbf{x},\mathbf{y}) \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{2} \partial X_{l}} \beta^{N}(\mathbf{x}) \overline{C}_{2}^{N} \\ \end{cases} \end{cases}$$

$$+ \left\{ \frac{\frac{\partial \beta^{N}(\mathbf{x})}{\partial X_{1}} u_{1}^{N}}{\frac{\partial \beta^{N}(\mathbf{x})}{\partial X_{2}} u_{1}^{N}} \right\} + \left\{ \frac{\frac{\partial \chi_{1}^{kl}(\mathbf{x},\mathbf{y})}{\partial Y_{1}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{l}} \beta^{N}(\mathbf{x}) \overline{C}_{1}^{N} \frac{\partial Y_{1}}{\partial X_{1}}}{\frac{\partial \chi_{1}^{kl}(\mathbf{x},\mathbf{y})}{\partial Y_{2}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{l}} \beta^{N}(\mathbf{x}) \overline{C}_{1}^{N} \frac{\partial Y_{2}}{\partial X_{2}}}{\frac{\partial \chi_{2}^{kl}(\mathbf{x},\mathbf{y})}{\partial Y_{1}} \frac{\partial u_{k}^{kl}(\mathbf{x})}{\partial X_{l}} \beta^{N}(\mathbf{x}) \overline{C}_{2}^{N} \frac{\partial Y_{2}}{\partial X_{2}}} \right\}$$
(3.17)

Let 
$$[A] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$
.

Then the equation (3.17) can be written as shown below:

$$\begin{cases} e_{11} \\ e_{22} \\ e_{12} \end{cases} = \begin{bmatrix} A \end{bmatrix} \mathbf{x} \begin{bmatrix} B_1 \end{bmatrix} \mathbf{x} \begin{bmatrix} B_2 \end{bmatrix} \mathbf{x} \begin{bmatrix} B_3 \end{bmatrix} \mathbf{x} \begin{bmatrix} B_4 \end{bmatrix} \mathbf{x} \begin{bmatrix} B_5 \end{bmatrix} \mathbf{x} \begin{bmatrix} B_6 \end{bmatrix} \mathbf{x} \begin{bmatrix} B_7 \end{bmatrix} \mathbf{x} \begin{bmatrix} B_8 \end{bmatrix}$$
(3.18)

where

$$\begin{bmatrix} B_1 \end{bmatrix} = \begin{cases} u_1^H(\mathbf{x}) \frac{\partial \beta^N(\mathbf{x})}{\partial X_1} C_1^N \\ u_1^H(\mathbf{x}) \frac{\partial \beta^N(\mathbf{x})}{\partial X_2} C_1^N \\ u_2^H(\mathbf{x}) \frac{\partial \beta^N(\mathbf{x})}{\partial X_1} C_2^N \\ u_2^H(\mathbf{x}) \frac{\partial \beta^N(\mathbf{x})}{\partial X_2} C_2^N \end{cases}$$
(3.19)

$$\begin{bmatrix} B_2 \end{bmatrix} = -\begin{cases} u_1^H(\mathbf{x}_{(N)}) \frac{\partial \beta^N(\mathbf{x})}{\partial X_1} C_1^N \\ u_1^H(\mathbf{x}_{(N)}) \frac{\partial \beta^N(\mathbf{x})}{\partial X_2} C_1^N \\ u_2^H(\mathbf{x}_{(N)}) \frac{\partial \beta^N(\mathbf{x})}{\partial X_1} C_2^N \\ u_2^H(\mathbf{x}_{(N)}) \frac{\partial \beta^N(\mathbf{x})}{\partial X_2} C_2^N \end{cases}$$
(3.20)

$$\begin{bmatrix} B_3 \end{bmatrix} = \begin{cases} \frac{\partial u_1^H(\mathbf{x})}{\partial X_1} \beta^N(\mathbf{x}) C_1^N \\ \frac{\partial u_1^H(\mathbf{x})}{\partial X_2} \beta^N(\mathbf{x}) C_1^N \\ \frac{\partial u_2^H(\mathbf{x})}{\partial X_1} \beta^N(\mathbf{x}) C_2^N \\ \frac{\partial u_2^H(\mathbf{x})}{\partial X_1} \beta^N(\mathbf{x}) C_2^N \end{cases}$$
(3.21)

$$\begin{bmatrix} B_{4} \end{bmatrix} = \begin{cases} \chi_{1}^{kl}(\mathbf{x}, \mathbf{y}) \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{1}} \frac{\partial \beta^{N}(\mathbf{x})}{\partial X_{1}} \overline{C}_{1}^{N} \\ \chi_{1}^{kl}(\mathbf{x}, \mathbf{y}) \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{1}} \frac{\partial \beta^{N}(\mathbf{x})}{\partial X_{2}} \overline{C}_{1}^{N} \\ \chi_{2}^{kl}(\mathbf{x}, \mathbf{y}) \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{1}} \frac{\partial \beta^{N}(\mathbf{x})}{\partial X_{2}} \overline{C}_{2}^{N} \\ \chi_{2}^{kl}(\mathbf{x}, \mathbf{y}) \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{1}} \frac{\partial \beta^{N}(\mathbf{x})}{\partial X_{2}} \overline{C}_{2}^{N} \end{bmatrix}$$

$$\begin{bmatrix} B_{5} \end{bmatrix} = \begin{cases} \frac{\partial \chi_{1}^{kl}(\mathbf{x}, \mathbf{y})}{\partial X_{1}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{1}} \frac{\partial \beta^{N}(\mathbf{x})}{\partial X_{2}} \overline{C}_{2}^{N} \\ \frac{\partial \chi_{1}^{kl}(\mathbf{x}, \mathbf{y})}{\partial X_{2}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{1}} \beta^{N}(\mathbf{x}) \overline{C}_{1}^{N} \\ \frac{\partial \chi_{2}^{kl}(\mathbf{x}, \mathbf{y})}{\partial X_{2}} \frac{\partial u_{k}^{k}(\mathbf{x})}{\partial X_{1}} \beta^{N}(\mathbf{x}) \overline{C}_{2}^{N} \\ \frac{\partial \chi_{2}^{kl}(\mathbf{x}, \mathbf{y})}{\partial X_{2}} \frac{\partial u_{k}^{k}(\mathbf{x})}{\partial X_{1}} \beta^{N}(\mathbf{x}) \overline{C}_{2}^{N} \\ \frac{\partial \chi_{2}^{kl}(\mathbf{x}, \mathbf{y})}{\partial X_{2}} \frac{\partial u_{k}^{k}(\mathbf{x})}{\partial X_{1}} \beta^{N}(\mathbf{x}) \overline{C}_{2}^{N} \\ \frac{\partial \chi_{2}^{kl}(\mathbf{x}, \mathbf{y})}{\partial X_{2}} \frac{\partial u_{k}^{k}(\mathbf{x})}{\partial X_{1}} \beta^{N}(\mathbf{x}) \overline{C}_{2}^{N} \\ \chi_{1}^{kl}(\mathbf{x}, \mathbf{y}) \frac{\partial^{2} u_{k}^{k}(\mathbf{x})}{\partial X_{1} \partial X_{1}} \beta^{N}(\mathbf{x}) \overline{C}_{2}^{N} \\ \chi_{2}^{kl}(\mathbf{x}, \mathbf{y}) \frac{\partial^{2} u_{k}^{k}(\mathbf{x})}{\partial X_{1} \partial X_{1}} \beta^{N}(\mathbf{x}) \overline{C}_{2}^{N} \\ \chi_{2}^{kl}(\mathbf{x}, \mathbf{y}) \frac{\partial^{2} u_{k}^{k}(\mathbf{x})}{\partial X_{1} \partial X_{1}} \beta^{N}(\mathbf{x}) \overline{C}_{2}^{N} \\ \chi_{2}^{kl}(\mathbf{x}, \mathbf{y}) \frac{\partial^{2} u_{k}^{k}(\mathbf{x})}{\partial X_{1} \partial X_{1}} \beta^{N}(\mathbf{x}) \overline{C}_{2}^{N} \\ \chi_{2}^{kl}(\mathbf{x}, \mathbf{y}) \frac{\partial^{2} u_{k}^{k}(\mathbf{x})}{\partial X_{2} \partial X_{1}} \beta^{N}(\mathbf{x}) \overline{C}_{2}^{N} \\ \chi_{2}^{kl}(\mathbf{x}, \mathbf{y}) \frac{\partial^{2} u_{k}^{H}(\mathbf{x})}{\partial X_{2} \partial X_{1}} \beta^{N}(\mathbf{x}) \overline{C}_{2}^{N} \\ \end{pmatrix}$$
(3.24)

$$\begin{bmatrix} B_7 \end{bmatrix} = \begin{cases} \frac{\partial \beta^N(\mathbf{x})}{\partial X_1} u_1^N \\ \frac{\partial \beta^N(\mathbf{x})}{\partial X_2} u_1^N \\ \frac{\partial \beta^N(\mathbf{x})}{\partial X_1} u_2^N \\ \frac{\partial \beta^N(\mathbf{x})}{\partial X_2} u_2^N \end{cases}$$
(3.25)

$$\begin{bmatrix} B_{8} \end{bmatrix} = \begin{cases} \frac{\partial \chi_{1}^{kl}(\mathbf{x}, \mathbf{y})}{\partial Y_{1}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{l}} \beta^{N}(\mathbf{x}) \overline{C}_{1}^{N} \frac{\partial Y_{1}}{\partial X_{1}} \\ \frac{\partial \chi_{1}^{kl}(\mathbf{x}, \mathbf{y})}{\partial Y_{2}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{l}} \beta^{N}(\mathbf{x}) \overline{C}_{1}^{N} \frac{\partial Y_{2}}{\partial X_{2}} \\ \frac{\partial \chi_{2}^{kl}(\mathbf{x}, \mathbf{y})}{\partial Y_{1}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{l}} \beta^{N}(\mathbf{x}) \overline{C}_{2}^{N} \frac{\partial Y_{1}}{\partial X_{1}} \\ \frac{\partial \chi_{2}^{kl}(\mathbf{x}, \mathbf{y})}{\partial Y_{2}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{l}} \beta^{N}(\mathbf{x}) \overline{C}_{2}^{N} \frac{\partial Y_{2}}{\partial X_{1}} \\ \frac{\partial \chi_{2}^{kl}(\mathbf{x}, \mathbf{y})}{\partial Y_{2}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{l}} \beta^{N}(\mathbf{x}) \overline{C}_{2}^{N} \frac{\partial Y_{2}}{\partial X_{2}} \end{cases}$$
(3.26)

## 3.3 Explicit Parameterization of the Cell Geometry over the Element

The equation (3.23) includes the term 
$$\frac{\partial \chi_1^{kl}(\mathbf{x}, \mathbf{y})}{\partial X_i} \frac{\partial u_k^H(\mathbf{x})}{\partial X_l}$$
, which is related

to the rate change in the microcell solution with respect to  $X_j$ . Changes in  $X_j$  are dependent on changes in the parameters,  $hx_2(\mathbf{x}, \mathbf{y})$  and  $hy_2(\mathbf{x}, \mathbf{y})$ , controlling the cell size. Essentially  $hx_2(\mathbf{x}, \mathbf{y})$  and  $hy_2(\mathbf{x}, \mathbf{y})$  define an explicit parameterization of the cell size in terms of macro element positions. The microcell approximation  $\chi_i^{kl}(\mathbf{x}, \mathbf{y})$ , is shown below:

$$\chi_i^{kl}(\mathbf{x}, \mathbf{y}) = \psi^N(\mathbf{x}, \mathbf{y})\chi_i^{kl N}(\mathbf{x})$$
(3.27)

Then evaluating equation (3.27) at all Gauss point *ij*, it can be seen that:

$$\chi_{i}^{kl}(\mathbf{x}, \mathbf{y})\Big|_{\mathbf{x}=x^{ij}} = \psi^{N}(\mathbf{x}, \mathbf{y})_{\mathbf{x}=x^{ij}} \chi_{i}^{kl N}(\mathbf{x})_{\mathbf{x}=x^{ij}}$$
$$= \psi^{N ij}(\mathbf{y}) \chi_{i}^{kl N}(\mathbf{x}^{ij})$$
$$= \chi_{i}^{kl ij}(\mathbf{y})$$
(3.28)

Thus,

$$\frac{\chi_{1}^{kl}(\mathbf{x},\mathbf{y})}{\partial X_{i}}\bigg|_{\mathbf{x}=x^{ij}} = \frac{\partial\chi_{1}^{kl\,ij}(\mathbf{y})}{\partial X_{i}}\bigg|_{\mathbf{x}=x^{ij}} = \frac{\partial}{\partial X_{i}} \left[ \psi^{N\,ij}(\mathbf{y})\chi_{i}^{kl\,N}(\mathbf{x}^{ij}) \right]$$
(3.29)

Or

$$\frac{\chi_1^{kl}(\mathbf{x},\mathbf{y})}{\partial X_i}\bigg|_{\mathbf{x}=x^{ij}} = \frac{\partial \psi^{N\,ij}(\mathbf{y})}{\partial X_i} \chi_i^{kl\,N}(\mathbf{x}^{ij}) + \psi^{N\,ij}(\mathbf{y}) \frac{\partial \chi_1^{kl\,N}(\mathbf{x}^{ij})}{\partial X_i}$$

$$= \left[\frac{\partial \psi^{Nij}(\mathbf{y})}{\partial hx_{2}(\mathbf{x}^{ij})}\frac{\partial hx_{2}(\mathbf{x}^{ij})}{\partial X_{i}} + \frac{\partial \psi^{Nij}(\mathbf{y})}{\partial hy_{2}(\mathbf{x}^{ij})}\frac{\partial hy_{2}(\mathbf{x}^{ij})}{\partial X_{i}}\right]\chi_{i}^{klN}(\mathbf{x})$$
$$+ \psi^{Nij}(\mathbf{y})\left[\frac{\partial \chi_{1}^{klN}(\mathbf{x})}{\partial hx_{2}(\mathbf{x}^{ij})}\frac{\partial hx_{2}(\mathbf{x}^{ij})}{\partial X_{i}} + \frac{\partial \chi_{1}^{klN}(\mathbf{x})}{\partial hy_{2}(\mathbf{x}^{ij})}\frac{\partial hy_{2}(\mathbf{x}^{ij})}{\partial X_{i}}\right]$$
(3.30)

See the Appendix B.1 (equations B.14 through B.31) for detail processing in the equation (3.23). The rate of changes in the cell solution with respect to  $\frac{\partial \chi_1^{kl \ N}(\mathbf{x})}{\partial h x_2(\mathbf{x}^{ij})}$  and  $\frac{\partial \chi_1^{kl \ N}(\mathbf{x})}{\partial h y_2(\mathbf{x}^{ij})}$  can be approximated using finite differences. For

example, if k = l = 1,

$$\frac{\partial \chi_1^{11N}(\mathbf{x})}{\partial h x_2(\mathbf{x})} = \frac{\chi_1^{11N}(\mathbf{x})_{h x_2(\mathbf{x}) + \Delta h x_2} - \chi_1^{11N}(\mathbf{x})_{h x_2(\mathbf{x})}}{\Delta h x_2}$$
(3.31)

$$\frac{\partial \chi_1^{11N}(\mathbf{x})}{\partial h y_2(\mathbf{x})} = \frac{\chi_1^{11N}(\mathbf{x})_{hy_2(\mathbf{x}) + \Delta h y_2} - \chi_1^{11N}(\mathbf{x})_{hy_2(\mathbf{x})}}{\Delta h y_2}$$
(3.32)

### 3.4 Coordinate Transformation

The equation (3.19) through equation (3.26) will be converted from the Cartesian coordinate system to the natural coordinate system (see Appendix B.1 for the detail presentation of the transformation process). Both the macro coordinate **x** and the micro coordinate **y** must be transformed. Thus the typical function  $f(\mathbf{x})$  in macro coordinate system and  $g(\mathbf{y})$  in micro coordinate system are changed to  $f(\overline{\xi})$  and  $g(\xi)$  respectively where  $\overline{\xi}$  is the natural coordinate for the macro system and  $\xi$  is the natural coordinate for the micro system. After each transformation has been completed, the following equations can be formed:

Transforming equation (3.19) - (3.26) gives,

$$\begin{bmatrix} B_1 \end{bmatrix} = \begin{bmatrix} u_N^H (\xi) \end{bmatrix}_{4X4} \begin{bmatrix} DJBX(\xi) \end{bmatrix}_{4x18} \begin{bmatrix} C_i^N \\ C_i^N \end{bmatrix}_{18x1}$$
(3.33)

$$\begin{bmatrix} B_2 \end{bmatrix} = \begin{bmatrix} DJBX(\xi) \\ \xi \end{bmatrix}_{4x18} \begin{bmatrix} u_i^{HN} \\ u_i^{RN} \end{bmatrix}_{18x18} \{ C_i^N \}_{18x1}$$
(3.34)

$$\begin{bmatrix} B_3 \end{bmatrix} = \begin{bmatrix} DJBX \begin{pmatrix} \Xi \\ \xi \end{pmatrix} \end{bmatrix}_{4x18} \begin{bmatrix} u_i^{HN} \end{bmatrix}_{18X2} \begin{bmatrix} \beta^N \begin{pmatrix} \Xi \\ \xi \end{pmatrix} \end{bmatrix}_{2x18} \{ C_i^N \}_{18X1}$$
(3.35)

$$[B_{4}] = \left[ \left( \left[ \beta(\xi) \right]_{2x18} \left[ \chi_{i}^{11^{ij}N} \right]_{18x3} \left[ A \right] \left[ DJBX(\xi) \right]_{4x4} \left[ u_{i}^{H} \right]_{18x1} \right) \right]_{4x4} \left[ DJB \right]_{4x18} \left\{ \overline{C}_{i}^{N} \right\}_{18x1}$$

$$(3.36)$$

$$[B_{5}] = \begin{cases} \left\{ \begin{bmatrix} \frac{\partial \psi^{1ij}}{\partial hx_{2}(\xi)} \frac{\partial hx_{2}(\xi)}{\partial X_{1}} \end{bmatrix}_{4X18} \begin{bmatrix} \chi_{i}^{kl N}(\xi) \end{bmatrix}_{18X3} \\ + \begin{bmatrix} \frac{\partial \psi^{1ij}}{\partial hx_{2}(\xi)} \frac{\partial hy_{2}(\xi)}{\partial X_{1}} \end{bmatrix}_{4X18} \begin{bmatrix} \chi_{i}^{kl N}(\xi) \end{bmatrix}_{18X3} \\ + \begin{bmatrix} \frac{\partial \psi^{1ij}}{\partial hy_{2}(\xi)} \frac{\partial hy_{2}(\xi)}{\partial X_{1}} \end{bmatrix}_{4X18} \begin{bmatrix} \chi_{i}^{kl N}(\xi) \end{bmatrix}_{18X3} \\ + \begin{bmatrix} \psi^{Nij}(\xi,\eta) \end{bmatrix}_{4X36} \begin{bmatrix} \frac{\partial \chi_{i}^{kl N}(\xi)}{\partial hx_{2}(\xi)} \frac{\partial hx_{2}(\xi)}{\partial X_{1}} \end{bmatrix}_{36X3} \\ + \begin{bmatrix} \psi^{Nij}(\xi,\eta) \end{bmatrix}_{4X36} \begin{bmatrix} \frac{\partial \chi_{i}^{kl N}(\xi)}{\partial hy_{2}(\xi)} \frac{\partial hy_{2}(\xi)}{\partial X_{1}} \end{bmatrix}_{36X3} \\ + \begin{bmatrix} \psi^{Nij}(\xi,\eta) \end{bmatrix}_{4X36} \begin{bmatrix} \frac{\partial \chi_{i}^{kl N}(\xi)}{\partial hy_{2}(\xi)} \frac{\partial hy_{2}(\xi)}{\partial X_{1}} \end{bmatrix}_{36X3} \end{cases} \begin{bmatrix} A \end{bmatrix} DJBX(\xi) \\ \begin{bmatrix} \mu^{N}(\xi) \end{bmatrix}_{2x18} \begin{bmatrix} \xi^{N}(\xi) \end{bmatrix}_{x18} \end{bmatrix}_{x18} \begin{bmatrix} \xi^{N}(\xi) \end{bmatrix}_{x18} \begin{bmatrix} \xi^{N}(\xi) \end{bmatrix}_{x18} \begin{bmatrix} \xi^{N}(\xi) \end{bmatrix}_{x18} \begin{bmatrix} \xi^{N}(\xi) \end{bmatrix}_{x18} \end{bmatrix}_{x18} \begin{bmatrix} \xi^{N}(\xi) \end{bmatrix}_{x18} \begin{bmatrix} \xi^{N}(\xi) \end{bmatrix}_{x18} \end{bmatrix}_{x18} \begin{bmatrix} \xi^{N}(\xi) \end{bmatrix}_{x18} \begin{bmatrix} \xi^{N}(\xi) \end{bmatrix}_{x18} \end{bmatrix}_{x18} \end{bmatrix}_{x18} \end{bmatrix}_{x18} \begin{bmatrix} \xi^$$

$$\left[ B_{6} \right] = \begin{bmatrix} \left[ \left[ \beta(\xi) \right]_{2x18} \left[ \chi_{i}^{11^{ij}N} \right]_{18x3} \left[ A \right] \right] \left[ \left[ J \right]^{-1} \right]_{4x4} \left\{ \begin{array}{c} \Gamma_{11} \left[ \frac{\partial^{2}}{\partial \xi_{i} \partial \xi_{j}} \right]_{4x2} \left[ \beta^{N}(\frac{e}{\xi}) \right]_{2x18} \\ + \Gamma_{12} \left[ \frac{\partial^{2}}{\partial \xi_{j} \partial \xi_{i}} \right]_{4x2} \left[ \beta^{N}(\frac{e}{\xi}) \right]_{2x18} \right] \left[ \mu^{H} \right]_{18x1} \\ \left[ \left[ \beta(\xi) \right]_{2x18} \left[ \chi_{i}^{11^{ij}N} \right]_{18x3} \left[ A \right] \right] \left[ \left[ J \right]^{-1} \right]_{4x4} \left\{ \begin{array}{c} \Gamma_{11} \left[ \frac{\partial^{2}}{\partial \xi_{i} \partial \xi_{j}} \right]_{4x2} \left[ \beta^{N}(\frac{e}{\xi}) \right]_{2x18} \right] \left[ \mu^{H} \right]_{18x1} \\ + \Gamma_{22} \left[ \frac{\partial^{2}}{\partial \xi_{i} \partial \xi_{j}} \right]_{4x2} \left[ \beta^{N}(\frac{e}{\xi}) \right]_{2x18} \\ + \Gamma_{22} \left[ \frac{\partial^{2}}{\partial \xi_{i} \partial \xi_{i}} \right]_{4x2} \left[ \beta^{N}(\frac{e}{\xi}) \right]_{2x18} \right] \left[ \mu^{H} \right]_{18x1} \\ \left\{ \mu^{H} \right\}_{18x1} \\ + \Gamma_{22} \left[ \frac{\partial^{2}}{\partial \xi_{i} \partial \xi_{i}} \right]_{4x2} \left[ \beta^{N}(\frac{e}{\xi}) \right]_{2x18} \right] \left\{ \mu^{H} \right\}_{18x1} \\ \left\{ \mu^{H} \right\}_{18x2} \\ \left\{ \mu^{H} \right$$

$$[B_7] = [DJBX]_{4X18} \{u_i^N\}_{18X1}$$
(3.39)

$$\begin{bmatrix} B_8 \end{bmatrix} = \left[ \left\{ \left[ \left[ DJBY \right]_{4X18} \left[ \chi_1^{kl N} (\mathbf{x}^{ij}) \right]_{18X3} \right] \left[ A \right] \left[ DJBX \right]_{4X18} \left\{ u_i^H \right\}_{18X1} \right\} \left\{ \frac{\partial Y_i}{\partial X_i} \right\}_{4X1} \right] \beta \left[ \xi \right]_{2X18} \left[ \zeta \right]_{2X1$$

Introducing equation (3.33) - (3.40), the equation (3.18) can be evaluated. Arranging the equation (3.18) in blocks corresponding to the correction coefficients ( $\mathbf{C}$  and  $\overline{\mathbf{C}}$ ) and displacement  $\mathbf{u}$ , the following equation is obtain:

$$\begin{cases} e_{11} \\ e_{22} \\ e_{12} \end{cases} = [P]_{3x54} \times \{UC\}_{54x1}$$
(3.41)

where,

$$[P]_{3x54} = \left[ [\text{term of coeff. } \mathbf{C}] \right] \left[ \text{term of coeff. } \mathbf{\overline{C}} \right] \left[ \text{term of coeff. } \mathbf{u} \right]_{3x54}$$
(3.42)

$$\{UC\}_{54x1} = \left[C_1^{\ 1}C_2^{\ 1}...C_1^{\ N}C_2^{\ N} \mid \overline{C}_1^{\ 1}\overline{C}_2^{\ 1}...\overline{C}_1^{\ N}\overline{C}_2^{\ N} \mid u_1^{\ 1}u_2^{\ 1}...u_1^{\ N}u_2^{\ N}\right]^T (3.43)$$
## 3.5 NPH Element Stiffness Matrix

The potential energy in a linear elastic body is defined as follows:

$$\Pi_{\mathrm{P}} = \int_{V} \left( \frac{1}{2} \{ \mathbf{e} \}^{T} [D] \{ \mathbf{e} \} - \{ \mathbf{e} \}^{T} [D] \{ \mathbf{e}_{0} \} + \{ \mathbf{e} \}^{T} \{ \sigma_{0} \} \right) dV$$
$$- \int_{V} \{ \mathbf{u} \}^{T} \{ \overline{\mathbf{F}} \} dV - \int_{S} \{ \mathbf{u} \}^{T} \{ \Phi \} dS - \{ \mathbf{U} \} \{ F \}$$
(3.44)

where

$$\{\mathbf{e}\} = \begin{cases} e_{11} \\ e_{22} \\ e_{12} \end{cases}$$
 is the strain field

[D] = Material property matrix

 $\{\mathbf{e}_0\}$  and  $\{\sigma_0\}$  = Initial strain and initial stress

$$\{\mathbf{u}\} = \begin{cases} u_1 \\ u_2 \end{cases}$$
, Displacement field

$$\{\overline{\mathbf{F}}\}$$
 = Body force

 $\{\Phi\}$  = surface traction

 $\{\mathbf{U}\}$  = Displacement vector at points where concentration forces are applied

 $\{\!\mathbf{F}\!\}$  = Concentrated loads applied to particular degrees-of-freedom

From the equation (3.44), the multi-scale variational model for the potential energy in a linear elastic body can be obtained by an averaging process,

$$\overline{\Pi}_{P} = \int_{V} \frac{1}{A} \int_{A} \left\{ \frac{1}{2} \left\{ \mathbf{e} \right\}^{T} [D] \left\{ \mathbf{e} \right\} - \left\{ \mathbf{e} \right\}^{T} [D] \left\{ \mathbf{e}_{0} \right\} + \left\{ \mathbf{e} \right\}^{T} \left\{ \sigma_{0} \right\} \right\} dA dV$$
$$- \int_{V} \left\{ \mathbf{u} \right\}^{T} \left\{ \overline{\mathbf{F}} \right\} dV - \int_{S} \left\{ \mathbf{u} \right\}^{T} \left\{ \Phi \right\} dS - \left\{ \mathbf{U} \right\} \left\{ F \right\}$$
(3.45)

where, A is the microscopic cell.

 $\frac{1}{A} \int_{Area} dA$  is the "average over" the microstructure cell of the strain energy

density at the particular macro position in  $\mathbf{x}$ . Substituting equation (3.41) into equation (3.45) and assuming that there is no initial strain, initial stress, body force or surface traction, the following equation can be obtained:

$$\overline{\Pi}_{P} = \frac{1}{2} \int_{V} \frac{1}{A} \int_{A} \left\{ \left\{ UC \right\}_{n}^{T} \left[ P \right]_{n}^{ijT} \left[ D \right] \left[ P \right]_{n}^{ij} \left\{ UC \right\}_{n} \right\} dAdV - \left\{ \mathbf{U} \right\} \left\{ F \right\}$$
(3.46)

$$\overline{\Pi}_{\mathrm{P}} = \frac{1}{2} \{ \mathbf{U} \mathbf{C} \}^{T} [K] \{ \mathbf{U} \mathbf{C} \} - \{ \mathbf{U} \} \{ \mathbf{F} \}$$
(3.47)

where,

$$\begin{bmatrix} K \end{bmatrix} = \int_{V} \frac{1}{A} \int_{A} \left\{ \left[ P \right]_{n}^{ij^{T}} \left[ D \right] \left[ P \right]_{n}^{ij} \right\} dA dV = \sum_{i=1}^{\#of \ elem} \left[ k \right]_{i}$$
(3.48)

$$\frac{\partial \overline{\Pi}_{p}}{\partial \{\mathbf{UC}\}} = 0 \text{ yields}$$
(3.49)

 $[K]{\mathbf{UC}} = {\mathbf{F}}$ (3.50)

59

Then, by using the Gauss Quadrature method, the equation (3.48) can be formed at the particular *ij* in microcell structure. Thus, the element stiffness matrix is shown:

$$\sum_{i=1}^{\#of elem} \sum_{i=1}^{N} \left[k\right]_{i} = \int_{V} \frac{1}{A} \int_{A} \left\{P\right]_{n}^{ij} \left[D_{Vp}\right] \left[P\right]_{n}^{ij} \right\} dAdV$$
$$= \sum_{i=1}^{N_{x}} \sum_{j=1}^{N_{y}} WT_{i} WT_{j} \left\{\frac{1}{A} \int_{A} \left\{P\right]_{n}^{ij} \left[D_{Vp}\right] \left[P\right]_{n}^{ij} \right\} t * DetJac^{ij}$$
$$= \sum_{i=1}^{N_{x}} \sum_{j=1}^{N_{y}} WT_{i} WT_{j} \left\{Micro^{ij}\right\} t * DetJac^{ij}$$
(3.51)

where,  $Micro^{ij} = \frac{1}{A} \int_{A} \left\{ P \right\}_{n}^{ij^{T}} \left[ D_{Vp} \right] \left[ P \right]_{n}^{ij} \right\}$ 

$$= \frac{1}{A_{y}^{ij}} \int_{0}^{L_{y}^{ij}} \int_{0}^{L_{y}^{ij}} \left[ \left[ P \right]^{T} \left[ D_{V_{p}} \right] \left[ P \right] \right]^{ij} thk \, dy_{1} dy_{2}$$

$$= \frac{1}{A_{y}^{ij}} \int_{-1-1}^{1} \left[ \left[ P \right]^{T} \left[ D_{V_{p}} \right] \left[ P \right] \right]^{ij} DetJac \ thk \, d\xi_{1} d\xi_{2}$$

$$= \frac{1}{A_{y}^{ij}} \sum_{ii=1}^{M_{x}} \sum_{jj=1}^{M_{y}} WT_{ii} WT_{jj} \left[ \left[ P \right]^{T} \left[ D_{V_{p}} \right] \left[ P \right] \right]^{ij} thk * DetJac$$
(3.52)

where, WT = Gauss Quadrature weight factor

*DetJac* = Determinant of Jacobian matrix

*thk* = Cell solution thickness

- t = Macro structure thickness
- $D_{v_p}$  = Variable material property in the micro cell

Also, *Micro<sup>ij</sup>* can be determined while the microcell structure changes depending on the  $hx_2(\xi)$  and  $hy_2(\xi)$  values at the each Gauss point. Finally, the NPH stiffness matrix can be formed as shown below:

$$\sum_{i=1}^{\#of\ elem} \left[k\right]_{i} = \sum_{i=1}^{N_{x}} \sum_{j=1}^{N_{y}} WT_{i}\ WT_{j}\left\{Micro^{ij}\right\}\ t * DetJac^{ij}$$
(3.53)

where

$$Micro^{ij} = \frac{1}{A_{y}^{ij}} \sum_{ii=1}^{M_{x}} \sum_{jj=1}^{M_{y}} WT_{ii}WT_{jj} \left( \left[ P \right]^{T} \left[ D_{Vp} \right] \right)^{ij} thk * DetJac$$

## 3.6 NPH Element Stress Calculation

The stress and strain relationship in a linear elastic body with an initial strain and a stress term is defined as follows:

$$\boldsymbol{\sigma} = \mathbf{E}(\mathbf{e} - \mathbf{e}_0) + \boldsymbol{\sigma}_0 \tag{3.54}$$

where,

$$\{\mathbf{e}_0\}$$
 = Initial strain  
 $\{\sigma_0\}$  = Initial stress

Since there is no initial strain and stress are presented, the equation (3.54) can be written in terms of the elastic constant and strain. The strain term was defined in the equation (3.41) which is a product of the [*P*] and nodal displacement, {*UC*}. Thus, the equation (3.54) can be written as below:

$$\boldsymbol{\sigma} = \mathbf{E}\mathbf{e} \tag{3.55}$$

$$\{\sigma\}_{3x1} = [E]_{3x3} [P]_{3x54} \{UC\}_{elemt}$$
(3.56)

where

$$[P]_{3x54} = \left[ \left[ \mathbf{C} \right]_{3X18} | \left[ \overline{\mathbf{C}} \right]_{3X18} | \left[ \mathbf{u} \right]_{3X18} \right]_{3x54}$$

At the microcell structure level shown in Figure 3.2, the stresses were computed based on the material properties at the location of the micro structural with the matrix  $[P_i]$ . Therefore, the equation 3.56 can be written as shown below:

$$\{\sigma_{1 \text{ of } 16}\}_{3x1} = [E_1]_{3x3} [P_1]_{3x54} \{UC\}_{elemt}$$
(3.57)

$$\{\sigma_{2 \text{ of } 16}\}_{3x1} = [E_2]_{3x3} [P_2]_{3x54} \{UC\}_{elemt}$$
(3.58)



Figure 3.2 Nodal Displacement at the Mesh Element and Microstructure: i.e.)  $E_1$ = 10 and  $E_2$  = 1000

## 3.7 Typical Allowable NPH Microcell Geometries

A genetic microcell structure in 2-D is shown in Figure 3.3. In the research proposal, the fiber size  $hx_1$  and  $hy_1$  are constant values and matrix sizes  $hx_2$  and  $hy_2$  are variable sizes in the micro coordinate **y** ( $Y_1$ ,  $Y_2$ ). Therefore, overall microcell structure will vary depending on the location of the integration points in the macro mesh.



Figure 3.3 A Microcell Structure in 2-D (Shaded areas Represent Fibers,  $hx_1$  and  $hy_1$ )

Various microcell structures can be created in order to fit the best characteristic structure of the Functionally Graded Materials. For example, in order to create the dot fiber micro structure as shown in Figure 3.4 (b), change the vertical and horizontal fiber material properties from the genetic microcell structure pictured in Figure 3.3 to the matrix material properties except the center location. Various types of the microcell structure for the Functionally Graded Materials can be generated as shown in Figure 3.4.



Figure 3.4 Variation of 2-D Microcell Structures with Fiber (shaded area): (a) Vertical Direction; (b) Center; (c) Outer Edges; (d) Distributed in the Center and Four Corners.

# 3.8 Data Collection Points for $hx_2$ and $hy_2$ Matrix Values in the Nonperiodic Microstructure

In order to characterize the nonperiodic microstructure pattern of the Functionally Graded Materials, the data collection point method was used. The meaning of the "data collection point" is to identify the variation of  $hx_2$  matrix values in 1-D or the variation of  $hx_2$  and  $hy_2$  matrix values in 2-D cases throughout the macro structure. The structure can have two or more data collection points per finite element. Figure 3.5 shows an example of the three data collection points in 1-D finite element case. Then, these data collection points were interpolated at the Gauss location. In order to increase the accuracy, a total of four integration points per finite element were used. These data collection points could be obtained manually or automatically and placed in the data input file for the NPH computation.



Figure 3.5 A Total of Three  $hx_2$  Data Collection Points per One Finite Element: Black Dots indicate Data Collection Points and Circles with Cross indicate the Location of Gauss Points.

Figure 3.6 shows the location of the data collection points of  $hx_2$  and  $hy_2$  in 2-D case. A total of nine data collection sets were obtained at the near of the element nodal points. As in the 1-D case, these data collection sets were interpolated at the Gauss integration points.



Figure 3.6 A Total of Nine Data Collection Points per Finite Element: Black Dots indicate Data Collection Points and Circles with Crosses indicate the Location of Gauss Points.

# Chapter 4

# Numerical Examples of NPH 1-D Cases

## 4.1 Introduction

The intent of this chapter is to test the performance of the nonperiodic analysis procedure on one dimension problem for which as exact solution exists. Six independent cases are considers including:

- Case 1 Comparison between the NPH and the Homogenization Solution
- Case 2 Descending Low Density Microcell Structures
- Case 3 Descending High Density Microcell Structure
- Case 4 Descending and Ascending Microcell Structure
- Case 5 Descending Microcell Structure with a Sudden Jump
- Case 6 Rapidly Varying Descending, Ascending and Descending

Microcell Structures.

# 4.2 Case 1 – Comparison between the NPH and the Homogenization Solutions

#### 4.2.1 Geometry Modeling

A structure which contains periodic microcells was created in order to compare the deformation results from the NPH and the homogenization methods. The cell structure possesses typical matrix and fiber configurations. The test configuration is pictured in Figure 4.1. In the case study, the length of the matrix component  $hx_2$  and the length of the fiber component  $hx_1$  were defined as 0.375 and 0.25, respectively. The Young's modulus of the matrix and fiber were assumed to be  $E_{matrix} = 10$  and  $E_{fiber} = 1000$ . A concentrated force, F was applied on the free end, and its value was F = 3.0. The total length of the structure was 2.0, and the cross section area of the structure was also 2.0. In the NPH algorithm, a single macro finite element was used for the numerical computations. Figure 4.2 showing the single macro element in the macro model for the NPH method and Figure 4.3 showing detailed view of the microcell structure, cell 1, for exact solutions.



Figure 4.1 A Periodic Structure with Microcells



Figure 4.2 A Single Macro Element for the NPH Method



Figure 4.3 Detailed View of the Cell 1 for the Exact Solutions:  $hx_2 = 0.375$  and  $hx_1 = 0.250$ 

## 4.2.2 Local and Global Deformation Analysis

The deformation results were obtained at the three different locations: X = 0.0, X = 1.0 and X = 2.0 which are the node point locations for three nodes in the macro-model. As is shown in Table 4.1, the homogenization and the NPH results were identical. This is due to the fact that the nonperiodic terms in the NPH algorithm are zero for this case. The result confirms that the NPH algorithm produces the exact homogenization solution for periodic structures. Therefore, the NPH method works correctly for the periodic cases.

Location in X	Homogenization	NPH
0.0	0.0000E+00	0.0000E+00
1.0	0.1129E+00	0.1129E+00
2.0	0.2258E+00	0.2258E+00

Table 4.1 Deformation Results of NPH and Homogenization Methods

4.3 Case 2 – Descending Low Density Microcell Structure

A Typical nonperiodic low density microcell structure was created to investigate the accuracy of the NPH algorithm. Three different methods were used as follows:

- Method 1 mapping the geometry with a single finite element and using two  $hx_2$  data collection points per finite element, picture in Figure 4.4(a).
- Method 2 mapping the geometry with a single finite element mesh and using three *hx*<sub>2</sub> data collection points per finite element, picture in Figure 4.4(b).
- Method 3 mapping with the geometry with four finite elements and using three *hx*<sub>2</sub> data collection points per finite element, picture in Figure 4.4(c).



Figure 4.4 Three Different  $hx_2$  Data Collection Methods; Black Dots indicate Data Collection Points and Circles with Cross indicate the Location of Gauss Points.

#### 4.3.1 Geometry Modeling

Consider the structure picture in Figure 4.5. The configuration of the microcell 1 was the same as cell 1 in Figure 4.3. The length of the matrix  $hx_2$  is 0.375 and the length of the fiber  $hx_1$  is 0.25. Then, the microcell matrix  $hx_2$  values, for cell 2 and 3, were decreased as compare to cell 1. The values of  $hx_2$  for cell 2 and for cell 3 are 0.225 and 0.025, respectively. The fiber length was not changed and took the value as 0.25. All boundary conditions were as presented in section 4.2.1.



Figure 4.5 Nonperiodic Low Density Microcell Structure

#### 4.3.2 Local and Global Deformation Analysis

The deformation results, comparing the solution using method 1, 2 and 3, are displayed in Table 4.2. The results indicate that when more data collection points were employed in the finite elements, the accuracy of the solution was increased. However, adding more finite elements while collecting the using the same number of data points did not significantly improve the results. This

occurred because the density of the microcell structures in the geometry was too coarse. Figure 4.6 shows that the NPH results reasonably follow the exact solution when the number of the finite element is increased.

Computational Methods	X = 1.0	X = 2.0	Percent of the Exact Solution at $x = 2.0$
Exact Solution	0.11288	0.18613	-
Method 1 – 2 Collection points w/ 1 Finite Element	0.10662	0.16960	91.1
Method 2 – 3 Collection points w/ 1 Finite Element	0.11027	0.17505	94.1
Method 3 – 3 Collection points w/ 4 Finite Elements	0.14588	0.17257	92.7

Table 4.2 Deformation Results for the Descending Low Density Microcell Structure Case



Figure 4.6 Deformation Results for the Descending Low Density Microcell with Three Different Data Collection Methods

## 4.4 Case 3 – Descending High Density Microcell Structure

4.4.1 Geometry Modeling

From an application perspective the high density microstructure is very important. Thus, a nonperiodic high density microcell structure was analyzed to investigate the accuracy of the NPH algorithm for these problems. As it is shown in the Figure 4.7, the numbers of microcells were added in the geometry. Three different methods which were described in section 4.3 were tested: Method 1, Method 2 and Method 3.



Figure 4.7 Descending Microcell Structure Case with 10 percent Descending

The geometry contains a total of eighteen microcells. The configuration of the microcell 1 was set as followed; the length of the matrix  $hx_2$  is 0.090 and the length of the fiber  $hx_1$  is 0.0262. The length of the matrix value was reduced continuously by 10 percent from cell to cell, starting from the cell 1. Thus, the length of the matrix  $hx_2$  at the cell 18 is 0.015. The matrix size values for the

eighteen cells are presented in Table 4.3. The fiber dimension was kept constant,

which was 0.0262. All boundary conditions were as presented in section 4.2.1.

Cell No.	Matrix Size $hx_2$	Cell No.	Matrix Size $hx_2$	Cell No.	Matrix Size $hx_2$
1	0.0900	7	0.0478	13	0.0254
2	0.0810	8	0.0430	14	0.0229
3	0.0729	9	0.0387	15	0.0206
4	0.0656	10	0.0349	16	0.0185
5	0.0590	11	0.0314	17	0.0167
6	0.0531	12	0.0282	18	0.0150

Table 4.3 The Matrix Size Values for the Descending and Ascending Microcell Structure: Fiber size  $hx_1 = 0.0262$ 

#### 4.4.2 Local and Global Deformation Analysis

The calculated deformation values for method 1, 2 and 3 are presented in Table 4.4. It can be seen that as compared to Case 2 in section 4.3, more consistent convergence trends are evident, and this is due to the increase in the number of microcells in the model. For example, the Method 3 used the most refined model in terms of numbers of elements and data collection points and produced the best accuracy. It is clear that collecting more matrix size values  $hx_2$ , in each finite element, helped to increase the accuracy. And adding more finite

elements in the geometry also produces better results. In Figure 4.8 and 4.9, the displacement patterns for the three methods are pictured.

Computational Methods	x = 1.0	x = 2.0	Percent of The Exact Solution at $x = 2.0$
Exact Solution	0.12675	0.23018	-
Method 1 – 2 Collection points w/ 1 Finite Element	0.13675	0.22448	97.5
Method 2 – 3 Collection points w/ 1 Finite Element	0.13695	0.22474	97.6
Method 3 – 3 Collection points w/ 4 Finite Elements	0.12722	0.22766	98.9

Table 4.4 The Deformation Results of the High Density Microcell Structure with 10 percent Reduction



Figure 4.8 Deformation Results of Descending Microcell Structure in High Density Microcell with Three Different Data Collection Methods



Figure 4.9 Detailed View of the Figure 4.5 (Red Box Region)

#### 4.5 Case 4 – Descending and Ascending Microcell Structure

#### 4.5.1 Geometry Modeling

The NPH algorithm was tested for structure with continuously varying microcells with the cell size both descending and ascending in terms of matrix size  $hx_2$ . A test case was created as pictured in Figure 4.10. In this case again the cell size is slowly varying, while ascending and descending, in terms of the matrix size  $hx_2$ . The geometry contains a total of fifteen microcells. Four macro finite elements were used with three  $hx_2$  data collection points per finite element.

The first microcell in the geometry was identical to Case 3; the length of the matrix  $hx_2$  is 0.090 and the length of the fiber  $hx_1$  is 0.0262. Then, 10 percent reduction was made from the first microcell matrix value  $hx_2$ , until it reached approximately 3/4 of the geometry. After that, the matrix value was increased by 116 percent continuously until it reached the end of the structure. Thus, the matrix value  $hx_2$ , at the end of the structure became 0.0578. The fiber dimension was kept constant value which was 0.0262. The matrix size values for the fifteen cells are presented in Table 4.5. All boundary conditions were as presented in section 4.2.1.



Figure 4.10 High Density Microcell Structure Case with 10 percent Descending and 16 percent Ascending Matrix Sizes

Cell	Matrix Size	Cell	Matrix Size	Cell	Matrix Size
No.	$hx_2$	No.	$hx_2$	No.	$hx_2$
1	0.0900	6	0.0531	11	0.0314
2	0.0810	7	0.0478	12	0.0366
3	0.0729	8	0.0430	13	0.0426
4	0.0656	9	0.0387	14	0.0496
5	0.0590	10	0.0349	15	0.0578

Table 4.5 The Matrix Size Values for the Descending and Ascending Microcell Structure: Fiber size  $hx_1 = 0.0262$ 

#### 4.5.2 Local and Global Deformation Analysis

Table 4.6 compares the NPH deformation results at global coordinates X = 1.0 and X = 2.0 with exact solution and the homogenization solution. The homogenization solution was added in the table in order to show the non-linearity of the NPH solution. Even though only four macro finite elements were used in the NPH model, the deformation results are extremely accurate. In Figure 4.11 and 4.12, the deformation results for NPH are compared to the exact solution.

Computational Methods	x = 1.0	x = 2.0	Percent of The Exact Solution at $x = 2.0$
Exact Solution	0.12675	0.24183	-
3 samples w/ 4 Finites	0.12726	0.23948	99.0
Homogenization Soln	0.13116	0.26233	108.5

Table 4.6 Global Deformation Values in High Density Microcell Structure Case with 10 percent Descending and 16 percent Ascending Matrix Sizes



Figure 4.11 Deformation Results for High Density Descending and Ascending Microcell Structure



Figure 4.12 Detailed View of the Figure 4.5 (Red Box Region)

#### 4.6 Case 5 – Descending Microcell Structure with a Sudden Jump

4.6.1 Geometry Modeling

One of the FGMs characteristics is that the geometry has a gradual variation in space. However, in this case, a problem with a sudden jump of microcell structure geometry was created and investigated. The microstructural model is pictured in Figure 4.13. The geometry involved a total of fifteen microcells, and the model incorporated four macro finite elements and used three  $hx_2$  data collection points per finite element.

In the developed model, the microcell matrix size  $hx_2$  was decreased by 15 percent from cell to cell, starting from the left hand side and a sudden jump (approximately 5 times more than its neighbor matrix size) was generated at the middle of structure. After that, the matrix value  $hx_2$  was decreased again by 34 percent from cell to cell. The cell matrix size values are presented in Table 4.7. All other boundary conditions were as same as Case 1 in section 4.2.1.



Figure 4.13 Descending Micro Structure with a Sudden Jump

Cell No.	Matrix Size $hx_2$	Cell No.	Matrix Size $hx_2$	Cell No.	Matrix Size $hx_2$
1	0.0900	6	0.0399	11	0.0655
2	0.0765	7	0.0339	12	0.0433
3	0.0650	8	0.0289	13	0.0286
4	0.0553	9	0.1500	14	0.0189
5	0.0470	10	0.0992	15	0.0125

Table 4.7 The Matrix Size Values for the Descending Microcell Structure with a Sudden Jump: Fiber size  $hx_1 = 0.0194$ 

#### 4.6.2 Local and Global Deformation Analysis

Table 4.8 presents the displacement results for the case of geometry with a sudden jump. The comparisons, between the exact solution and the NPH solution were made at four different locations: X = 0.5, 1.0, 1.5 and 2.0. In Figure 4.14, the deformation patterns for the NPH method are compared to the exact solution. The exact solution clearly showed that there are two connected but independent curves: one from before the sudden jump and the other one after the sudden jump. Figure 4.14 shows that the NPH solution does not accurately follow the exact solution for the case with sudden jumps. The reason is that the NPH algorithm uses the homogenized displacement solution  $\mathbf{u}_i^H$ , for correcting its deformation values. Thus, if there is a sudden jump at the micro structural level, the mathematical correction cannot be accurately determined.

Computational Methods	X = 0.5	X = 1.0	X = 1.5	X = 2.0
Exact Solution	0.06954	0.13118	0.21585	0.25681
NPH Solution (3 samples w/ 4 Finites)	0.06895	0.11546	0.19124	0.23861

Table 4.8 Deformation Results for the High Density Descending Microcell Structure with a Sudden Jump



Figure 4.14 Deformation Results of the High Density Descending Micro Structure with a Sudden Jump

# 4.7 Case 6 - Rapidly Varying Descending, Ascending and

#### **Descending Microcell Structures**

#### 4.7.1 Geometry Modeling

The tests in this section were designed to explore the ramification of the sudden jump result. Consider a case in which the matrix size value  $hx_2$  is rapidly varying, but does not have a sudden jump. Thus, the matrix value  $hx_2$  was continuously decreased, increased and decreased in a smooth but rapidly varying way. This generates a smooth transition among the microcells. The geometry includes a total of fifteen microcells and is pictured in Figure 4.15. The model incorporates four finite elements with three  $hx_2$  data collection points per finite element.

The microcell was reduced by 30 percent continuously from the first microcell until it reached the middle of the structure. After that it was increased by 130 percent then decreased 25 percent until the end of the structure was reached. The associated matrix size  $hx_2$  parameters are presented in Table 4.9. All boundary conditions were as presented in section 4.2.1.



Figure 4.15 High Density Microcells with Rapidly Varying Descending, Ascending and Descending Structure

Cell	Matrix Size	Cell	Matrix Size	Cell	Matrix Size
INO.	$nx_2$	INO.	$nx_2$	INO.	$nx_2$
1	0.0900	6	0.0151	11	0.0187
2	0.0630	7	0.0197	12	0.0140
3	0.0441	8	0.0256	13	0.0105
4	0.0309	9	0.0332	14	0.0079
5	0.0216	10	0.0249	15	0.0059

Table 4.9 The Matrix Size Values for the Rapidly Varying Descending, Ascending and Descending Structure: Fiber size  $hx_1 = 0.070$ 

4.7.2 Local and Global Deformation Analysis

Table 4.10 shows the deformation results at four different locations: X = 0.49, 0.95, 1.43 and 1.90. Figure 4.16 compares the NPH solution and the exact solution. These results indicate that the NPH method can accurately follows the exact solution for the case in which the microstructure is rapidly varying. Again, the use of three data collection point method, in NPH algorithm, is validated.

Table 4.10 Deformation Results of the High Density Microcell with Rapidly Varying Descending, Ascending and Descending Microcell Structure

Computational Methods	x = 0.49	x = 0.95	x = 1.43	x = 1.90
Exact Solution	0.05273	0.08004	0.11148	0.12911
NPH Solution (3 samples w/ 4 Finites)	0.04838	0.07785	0.10779	0.12733



Figure 4.16 Deformation Results of the High Density Microcell with Rapidly Varying Descending, Ascending and Descending Structure

# Chapter 5

# Numerical Examples of NPH 2-D Cases

#### 5.1 Introduction

After carefully tested the performance of the NPH algorithm in nonperiodic 1-D cases, further studies were conducted in two dimensional cases. In this chapter, not only the global and local deformation values were determined, but also local Von-Mises stresses were computed. These results, which were computed by the NPH method, were compared with the results of commercially available Finite Element Analysis (FEA) software. Five independent 2-D FGMs cases were considered and they are:

- Case 7 Periodic Microstructure
- Case 8 Descending Horizontal Fiber Strips in One Direction
- Case 9 Descending and Ascending FGMs with Square Fibers
- Case 10 Descending and Symmetric Matrix Structure
- Case 11 Descending Horizontal and Vertical Fiber Strips

### 5.2 Case 7 – Periodic Microstructure

#### 5.2.1 Geometry Modeling

In order to validate the performance of the NPH method in 2-D cases, a structure with periodic microcells was created. The deformation results from the

NPH method and the homogenization method were then compared. The microstructural model is pictured in Figure 5.1. A total of sixteen macro elements were used to mesh the entire geometry in the NPH method. Nine data collection points were utilized per finite element.

The lengths of the matrix component,  $hx_2$  and  $hy_2$ , are 0.085 and the lengths of the fiber component  $hx_1$  and  $hy_1$  are 0.030 in both X<sub>1</sub> and X<sub>2</sub> directions. The geometry includes a total of 100 periodic microcells. Thus, the size of the geometry was 2.0 by 2.0 with the thickness 1.0. The Young's modulus of the matrix  $E_1$  and fiber  $E_2$  were assumed to be 10 and 1000, respectively. The Poisson ratio for both the matrix and the fiber was assumed to be 0.3. The distribution force F was applied at the free end of the structure in X<sub>1</sub> direction and its value was 3.0.



Figure 5.1 Periodic Microcell Geometry and Boundary Conditions

5.2.2 Local and Global Deformation Analysis

As it is shown in Figure 5.2, the deformation results were obtained at the three different locations:  $X_1$ = 1.0, 1.5 and 2.0. The homogenization and the NPH deformation values are presented in Table 5.1. The results indicate that there is no difference between the NPH method and the Homogenized method at the nodal points. Therefore, the NPH algorithm works correctly for the 2-D periodic case.



Figure 5.2 Homogenization and NPH Deformation Results for the Periodic Microstructure at  $X_1 = 1.0$ ,  $X_1 = 1.5$  and  $X_1 = 2.0$ 

	Deformation at $X_1 = 1.0$			
Location at X <sub>2</sub>	HOMOG	NPH		
2.00	8.917E-03	8.916E-03		
1.75	8.906E-03	8.905E-03		
1.50	8.899E-03	8.897E-03		
1.25	8.897E-03	8.894E-03		
1.00	8.896E-03	8.893E-03		
0.75	8.897E-03	8.894E-03		
0.50	8.899E-03	8.897E-03		
0.25	8.906E-03	8.905E-03		
0.00	8.917E-03	8.916E-03		

Table 5.1 Comparison of the Deformation between Homogenized Solution and NPH Solution @  $X_1$ =1.0, 1.5 & 2.0

	Deformation at $X_1 = 1.5$			
Location at X <sub>2</sub>	HOMOG	NPH		
2.00	1.337E-02	1.337E-02		
1.75	1.336E-02	1.336E-02		
1.50	1.336E-02	1.335E-02		
1.25	1.335E-02	1.335E-02		
1.00	1.335E-02	1.335E-02		
0.75	1.335E-02	1.335E-02		
0.50	1.336E-02	1.335E-02		
0.25	1.336E-02	1.336E-02		
0.00	1.337E-02	1.337E-02		

	Deformation at $X_1 = 2.0$	
Location at X <sub>2</sub>	HOMOG	NPH
2.00	1.782E-02	1.782E-02
1.75	1.782E-02	1.781E-02
1.50	1.781E-02	1.781E-02
1.25	1.781E-02	1.781E-02
1.00	1.781E-02	1.780E-02
0.75	1.781E-02	1.781E-02
0.50	1.781E-02	1.781E-02
0.25	1.782E-02	1.781E-02
0.00	1.782E-02	1.782E-02

#### 5.3 Case 8 – Descending Horizontal Fiber Strips in One Direction

5.3.1 Geometry Modeling

The high density microcell strip structure was created to evaluate the performance of the NPH algorithm in 2-D. In the structure, a total of twenty groups of cell strip were used. The matrix values were decreased in only  $X_2$  direction and the structure model is pictured in Figure 5.3. In the NPH method, a total of sixteen macro elements were used to compute the global deformation values.

As is shown in Figure 5.4, the matrix size  $hy_2$  value is 0.0850 at the cell strip 1 and there are imaginary lines which separate the cell groups. The matrix size was linearly reduced by 11 percent starting from the cell strip 1 to cell strip 20. Thus, the last cell matrix value became 0.0093. The matrix size values for the 20 cells are presented in Table 5.2. The material properties and the boundary conditions were as presented in section 5.2.1.



Figure 5.3 Descending Horizontal Fiber Strips Microcell Structure: (a) Descending Horizontal Fiber Strips in X<sub>2</sub> Direction, (b) 16 Macro Elements for NPH



Figure 5.4 Detailed View of the Microcell Strips: Fiber Value  $hy_1 = 0.030$  (Constant)

Cell No.	Matrix Size hy <sub>2</sub>	Cell No.	Matrix Size hy <sub>2</sub>
1	0.0850	11	0.0265
2	0.0757	12	0.0236
3	0.0673	13	0.0210
4	0.0599	14	0.0187
5	0.0533	15	0.0166
6	0.0475	16	0.0148
7	0.0422	17	0.0132
8	0.0376	18	0.0117
9	0.0335	19	0.0104
10	0.0298	20	0.0093

Table 5.2 Matrix Size Values for the Descending Horizontal Fiber Strips
5.3.2 Local and Global Deformation Analysis

Using the conventional FEA method, models for a high density structure are required to have massive numbers of macro elements. Figure 5.5 illustrates the characteristic of the macro mesh. Table 5.3 compares the model sizes associated with the conventional FEA method and the NPH method.

	FEA (ABAQUS) Method	NPH Method
Total Macro Elements	17,956	16
No. of Integration Points per Element	4 (2X2)	16 (4X4)
Element Type	Four (4) node Lagrange	Nine (9) node Lagrange
Nodal DOF/ Total DOF	2/18,225	6/486

Table 5.3 Summary of Model Sizes for Case 8



Figure 5.5 Conventional FEA Mesh and Boundary Conditions

The local and global deformation results for the FEA and the NPH approaches were compared and pictured in Figure 5.6. In the matrix region, the distributed force was directly acting on the edge of the matrix. It should be noted that in Figure 5.6 the NPH displacement at the macro-nodes are presented. This is because the nature of the materials involved a microstructural behavior is associated cell solution which only varied in the  $X_2$  direction. This is a characteristic of the homogenization method.



Figure 5.6 Local Deformation Results: the NPH Solution and the FEA Solution at  $X_1 = 1.995$ 

### 5.4 Case 9 – Descending and Ascending FGMs with Square Fibers

### 5.4.1 Geometry Modeling

The NPH algorithm was tested for the descending and ascending matrix case with square fibers. The test model is shown in Figure 5.7 (a). A total of the 225 microcell groups were presented in the model. The microcells were reduced by 10 percent in both  $X_1$  and  $X_2$  directions until the cells reached approximately 3/4 of the length. Then the microcells were increased by 117 percent until they reached the end of the structure.

To understand the details of the modeling, Figure 5.8 is included. In Figure 5.8, it can be seen that there are imaginary lines which separate the cell groups. The initial matrix dimension started with 0.090 for both  $hx_2$  and  $hy_2$  and the last matrix element value was 0.0578. For example, cell 2 has the matrix values  $hx_2 = 0.081$ ,  $hy_2 = 0.090$  and the fiber value  $hx_1 = hy_1 = 0.030$ . The matrix values for the X<sub>1</sub> and X<sub>2</sub> direction are presented in Table 5.4. The material properties and the boundary conditions were as presented in section 5.2.1.



Figure 5.7 Descending and Ascending Matrix with Square Fibers



Figure 5.8 Detailed View of the Square Fibers: Fiber Value  $hx_1 \& hy_1 = 0.030$  (Constant)

Cell No.	Matrix Size	Cell No.	Matrix Size
$(X_1$ - dir.)	$hx_2 / hy_2$	$(X_2 - dir.)$	$hx_2 / hy_2$
1	0.0900 / 0.0900	1	0.0900 / 0.0900
2	0.0810/ 0.0900	16	0.0900 / 0.0810
3	0.0729/ 0.0900	31	0.0900 / 0.0729
4	0.0656/ 0.0900	46	0.0900 / 0.0656
5	0.0590/ 0.0900	61	0.0900 / 0.0590
6	0.0531/ 0.0900	76	0.0900/0.0531
7	0.0478/ 0.0900	91	0.0900 / 0.0478
8	0.0430/ 0.0900	106	0.0900 / 0.0430
9	0.0387/ 0.0900	121	0.0900 / 0.0387
10	0.0349/ 0.0900	136	0.0900 / 0.0349
11	0.0314/ 0.0900	151	0.0900 / 0.0314
12	0.0366/ 0.0900	166	0.0900 / 0.0366
13	0.0426/ 0.0900	181	0.0900 / 0.0426
14	0.0496/ 0.0900	196	0.0900 / 0.0496
15	0.0578/ 0.0900	211	0.0900 / 0.0578

Table 5.4 Matrix Sizes in Descending and Ascending Square Fibers

5.4.2 Local and Global Deformation Analysis

In the conventional FEA program, the model required a total of 24,235 macro elements in order to obtain accurate results. In the same model, the NPH method used 16 macro elements to compute the local and global deformation values. Table 5.5 compares the model sizes for the FEA method and the NPH method. Also, the FEA mesh configuration is shown in Figure 5.9 and Figure 5.10. The deformation results of the conventional FEA method are pictured in Figure 5.11.

	FEA (ABAQUS) Method	NPH Method
Total Macro Elements	24,235	16
No. of Integration Points per Element	4 (2X2)	16 (4X4)
Element Type	Four (4) node Lagrange	Nine (9) node Lagrange
Nodal DOF/ Total DOF	2/49,084	6/486

Table 5.5 Summary of Model Sizes for Case 9



Figure 5.9 Conventional FEA Mesh Sizes and Boundary Conditions



Figure 5.10 Detailed View of the FEA Mesh Sizes and Boundary Conditions (Red Box Region)



Figure 5.11 Deformation Results Using the Conventional FEA Method

As shown in Figure 5.12, the deformation results for the conventional FEA method and the NPH method are compared at  $X_1 = 1.9427$ . The location is set because the cell solutions in the microstructure were computed adjacent to the square fiber in the NPH method. The NPH local and global deformation profiles accurately matched the FEA results. Both the FEA method and the NPH method produced less deformation in the area, which had more square fibers.



Figure 5.12 Deformation Results of the Conventional FEA Method and the NPH Method at  $X_1 = 1.9427$ 

## 5.5 Case 10 – Descending and Symmetric Matrix Structure

### 5.5.1 Geometry Modeling

The high density nonperiodic descending structure was created to evaluate the NPH method. In this case, fiber strips were included in both in the  $X_1$ direction and the  $X_2$  direction. As is shown in Figure 5.13, the vertical strips were gradually reduced until they reached the end of the structure. On the other hand, the horizontal strips also were reduced, but the reduction started from the center of the structure. Thus, the model has a symmetric condition about the horizontal centerline of the model. The material properties and the boundary conditions were as presented in section 5.2.1.



Figure 5.13 Descending Matrix Microstructures in  $X_1$  Direction and Symmetric condition in  $X_2$  Direction with Descending Microstructures

The detailed view of the structure is pictured in Figure 5.14. The microcells were constructed in the "Fiber-Matrix-Fiber" pattern in the X<sub>2</sub> direction. Using the imaginary lines between the vertical and the horizontal fibers, a total of 560 microcells were generated in the structure. Each microcell structure has a pair of the matrix sizes  $hx_2$  and  $hy_2$ : for example, the cell 3 has  $hx_2 = 0.0673$  and  $hy_2 = 0.0093$ . The fiber size  $hx_1$  and  $hy_1$  were constant as 0.030. The matrix sizes  $hx_2$  and the horizontal properties and the boundary conditions were as presented in section 5.2.1.



Figure 5.14 Detailed View of Figure 5.13 (Red Box Region)

Cell No.	Matrix Size	Cell No.	Matrix Size
$(X_1 - dir.)$	$hx_2 / hy_2$	$(X_2 - dir.)$	$hx_2 / hy_2$
1	0.0850 / 0.0093	1	0.0850 / 0.0093
2	0.0757 / 0.0093	21	0.0850 / 0.0104
3	0.0673 / 0.0093	41	0.0850 / 0.0117
4	0.0599 / 0.0093	61	0.0850/0.0132
5	0.0533 / 0.0093	81	0.0850 / 0.0148
6	0.0475 / 0.0093	101	0.0850 / 0.0166
7	0.0422 / 0.0093	121	0.0850 / 0.0187
8	0.0376 / 0.0093	141	0.0850 / 0.0210
9	0.0335 / 0.0093	161	0.0850 / 0.0236
10	0.0298 / 0.0093	181	0.0850 / 0.0265
11	0.0265 / 0.0093	201	0.0850 / 0.0298
12	0.0236 / 0.0093	221	0.0850 / 0.0335
13	0.0210 / 0.0093	241	0.0850 / 0.0376
14	0.0187 / 0.0093	261	0.0850 / 0.0422
15	0.0166 / 0.0093	281	0.0850 / 0.0422
16	0.0148 / 0.0093	301	0.0850 / 0.0376
17	0.0132 / 0.0093	321	0.0850 / 0.0376
18	0.0117 / 0.0093	341	0.0850 / 0.0298
19	0.0104 / 0.0093	361	0.0850 / 0.0265
20	0.0093 / 0.0093	381	0.0850 / 0.0236
21	n/a	401	0.0850 / 0.0210
22	n/a	421	0.0850 / 0.0187
23	n/a	441	0.0850 / 0.0166
24	n/a	461	0.0850 / 0.0148
25	n/a	481	0.0850/0.0132
26	n/a	501	0.0850/0.0117
27	n/a	521	0.0850 / 0.0104
28	n/a	541	0.0850 / 0.0093

Table 5.6 Matrix Sizes in Symmetric and Descending Case

5.5.2 Local and Global Deformation Analysis

The comparison of the model sizes between the conventional FEA method and the NPH method are shown in Table 5.7. The FEA method required an element number approximately a thousand times more than the NPH method; the FEA employed 18,900 elements versus 16 elements used in the NPH model.

	FEA (ABAQUS) Method	NPH Method
Total Macro Elements	18,900	16
No. of Integration Points per Element	4 (2 x 2)	16 (4 x 4)
Element Type	Four (4) node Lagrange	Nine (9) node Lagrange
Nodal DOF/ Total DOF	2/19176	6/486

Table 5.7 Summary of Model Sizes for Case 10

The local and the global deformation results of the NPH method and the FEA method are in Figure 5.15 and Figure 5.16. The NPH deformations accurately followed the FEA results, and the global deformation U(x) matched the fiber regions of the FEA curve. The deformation results of the entire structure are shown in Figure 5.17. The highest deformation occurred at the center of the geometry due to the lower density of horizontal fibers there.



Figure 5.15 Symmetric and Descending Structure: Deformation Displacement at  $X_1 = 1.995$ 



Figure 5.16 Detailed View of Symmetric and Descending Structure (Red Box Region)



Figure 5.17 Deformation Results of the FEA Method (ABAQUS)

## 5.5.3 Local Stress Analysis

For this case study, the Von-Mises stress calculation algorithm was added in the NPH procedure and the NPH stress results were compared with the results of the commercially available FEA program such as ABAQUS. Figure 5.18 presents the FEA stress contours based on computed nodal point stresses. The largest stresses occur in the horizontal fibers, not in the vertical fibers when the force distribution is parallel to the horizontal fibers. The maximum stress was located at the middle of the structure near the free edge and the Von-Mises stresses value was 5.811. A detailed view of this high stress region is pictured in Figure 5.19.



Figure 5.18 Von-Mises Stress Results from FEA (ABAQUS)



Figure 5.19 Detailed View of the High Stress Region (Red Box Region): Max. Stress = 5.811

NPH was used to compute the Von-Mises stress values in the same area which was identified by the FEA as the maximum stress region. NPH computed the stress values at the integration points rather than at the element nodal points. Thus, the FEA stress values have been converted to the integration points. The NPH algorithm used 16 integration points while the FEA used four integration points per element. Figure 5.20 and Figure 5.21 show the stress results. The summary of the Von-Mises stress results is shown in Table 5.8. It should be noted that the FEA stress value at the nodal points were higher than at the integration points.



Figure 5.20 Von-Mises Stress Results by NPH method (Red Box Region in Figure 5.19): Max. Stress Value = 7.473



Figure 5.21 Von-Mises Stress Results by Commercially Available FEA (ABAQUS): Max. Stress Value = 5.777

Table 5.8 Von-Mises Stress in Descending and Symmetric Case

FEA	FEA	NPH
Max. Von-Mises Stress	Max. Von-Mises Stress	Max. Von-Mises Stress
at Nodal Point	at Integration Point	at Integration Point
5.811	5.777	7.473

## 5.6 Case 11 – Descending Horizontal and Vertical Fiber Strips

5.6.1 Geometry Modeling

In this case, a high density nonperiodic structure was created to evaluate the NPH method. The structure has vertical and horizontal fiber strips which are linearly reduced in both  $X_1$  and  $X_2$  directions simultaneously. The model is pictured in Figure 5.22. A total of sixteen macro elements were used to mesh the entire geometry in the NPH method and nine matrix data collection points, per finite element, were used. The material properties and the boundary conditions were as presented in section 5.2.1.



Figure 5.22 Simultaneously Reducing Matrix Values in X1 & X2 Directions

The detailed view of the microcell structure is pictured in Figure 5.23. The structure contains twenty microcells in each direction and a total of 400 microcells were used. These microcells were constructed by using imaginary lines, which separate the nonperiodic matrix values. The structure was constructed in similar manner to that used in section 5.5.2. For example, Cell 21 has the matrix values  $hx_2 = 0.0850$  and  $hy_2 = 0.0757$ . The fiber values were kept at the constant value of 0.030. However, because of the symmetric condition requirement in the microcell solution, a half fiber value of 0.015 was used in X<sub>2</sub> direction. The matrix sizes  $hx_2$  and  $hy_2$  in the microcells are presented in Table 5.9.



Figure 5.23 Detailed View of the Microcell Structure

Cell No.	Matrix Size	Cell No.	Matrix Size
$(X_1- dir.)$	$hx_2 / hy_2$	(X <sub>2</sub> - dir.)	$hx_2 / hy_2$
1	0.0850 / 0.0850	1	0.0850 / 0.0850
2	0.0757 / 0.0850	21	0.0850 / 0.0757
3	0.0673 / 0.0850	41	0.0850 / 0.0673
4	0.0599 / 0.0850	61	0.0850 / 0.0599
5	0.0533 / 0.0850	81	0.0850 / 0.0533
6	0.0475 / 0.0850	101	0.0850 / 0.0475
7	0.0422 / 0.0850	121	0.0850 / 0.0422
8	0.0376 / 0.0850	141	0.0850 / 0.0376
9	0.0335 / 0.0850	161	0.0850 / 0.0335
10	0.0298 / 0.0850	181	0.0850 / 0.0298
11	0.0265 / 0.0850	201	0.0850 / 0.0265
12	0.0236 / 0.0850	221	0.0850 / 0.0236
13	0.0210 / 0.0850	241	0.0850 / 0.0210
14	0.0187 / 0.0850	261	0.0850 / 0.0187
15	0.0166 / 0.0850	281	0.0850 / 0.0166
16	0.0148 / 0.0850	301	0.0850 / 0.0148
17	0.0132 / 0.0850	321	0.0850 / 0.0132
18	0.0117 / 0.0850	341	0.0850 / 0.0117
19	0.0104 / 0.0850	361	0.0850 / 0.0104
20	0.0093 / 0.0850	381	0.0850 / 0.0093

Table 5.9 Matrix Sizes in Descending Horizontal and Vertical Strips Case

5.6.2 Local and Global Deformation Analysis

The model size comparison between the conventional FEA method and the NPH method are summarized in Table 5.10. The FEA employed a total of 18,900 elements for the structure and the NPH used a total of sixteen elements.

	FEA (ABAQUS) Method	NPH Method
Total Macro Elements	17,807	16
No. of Integration Points per Element	4 (2 x 2)	16 (4 x 4)
Element Type	Four (4) node Lagrange	Nine (9) node Lagrange
Nodal DOF/ Total DOF	2/18224	6/486

Table 5.10 Summary of Model Sizes for Case 11

The NPH and the FEA results for the local and the global deformation are shown in Figure 5.24. The material was stiffer at the top than the bottom of the geometry due to more stiff fibers present at the top. Thus, the gradual deformation results occurred from the top edge to the bottom edge of the structure. The deformation results of the NPH method were computed at  $X_1 = 1.995$  because the microcell solutions were computed at Gauss integration points which were adjacent to the vertical fiber in the microcell. The displacement results computed by the FEA method are shown in Figure 5.25.



Figure 5.24 Deformation Results of Descending Horizontal and Vertical Strips



Figure 5.25 FEA (ABAQUS) Deformation Result

5.6.3 Local Stress Analysis

The Von-Mises Stresses were computed by the FEA and the NPH methods. The overall stress contour is displayed in Figure 5.26. It can be seen that the highest stress occurred at the right and the bottom of the geometry: Maximum Stress = 11.35 at a nodal point. The detailed view of this particular location was shown in Figure 5.27. The highest stresses occurred at the intersection of the horizontal and vertical fibers when the force distribution was in action at the free edge and parallel to the horizontal fibers.



Figure 5.26 Von-Mises Stress Results from FEA (ABAQUS)



Figure 5.27 Detailed View of Figure 5.26 (Red Box Region): Maximum Stress Value is 11.35

The Von-Mises stress was computed using the NPH method. The stress values were computed at the integration points. Thus, as in section 5.5.2, FEA stress values, which were computed at the nodal points, have been converted to integration point values. In this model, the FEA method used a total of four integration points and the NPH method used a total of sixteen integration points. The comparison of the local stress values by the FEA and the NPH methods are presented in Figure 5.28 and Figure 5.29. The summary of the results is shown in Table 5.11.

Table 5.11 Von-Mises Stress in Horizontal and Vertical Strips

FEA (ABAQUS)	FEA (ABAQUS)	NPH
Max. Von-Mises Stress	Max. Von-Mises Stress	Max. Von-Mises Stress
at Nodal Point	at Integration Point	at Integration Point
11.35	12.11	13.86



Figure 5.28 Von-Mises Stress Results by the NPH Method at Integration Points: Maximum Stress = 13.86



Figure 5.29 Von-Mises Stress Results by FEA (ABAQUS) at Integration Points: Maximum Stress = 12.11

# Chapter 6

## Summary and Conclusions

## 6.1 Summary

Based on the new theory of coupling the micro-macro structure, a nonperiodic homogenized (NPH) algorithm has been developed to solve complex Functionally Graded Materials (FGMs) problems. In order to verify the performance and the accuracy of the NPH algorithm, local deformation values, global deformation values and the Von-Mises stresses were computed. The results from the NPH method were compared with the exact solutions in 1-D cases and compared with commercially available FEA software results in 2-D cases. A total of eleven independent FGMs cases were investigated: 6 cases with 1-D problems and 5 cases with 2-D problems. And they are as follows:

## (A) 1-D cases for FGMs:

- Case 1 Comparison between the NPH and the Homogenization Solution
- Case 2 Descending Low Density Microcell Structures
- Case 3 Descending High Density Microcell Structure
- Case 4 Descending and Ascending Microcell Structure
- Case 5 Descending Microcell Structure with a Sudden Jump

 Case 6 – Rapidly Varying Descending, Ascending and Descending Microcell Structures.

#### (B) 2-D cases for FGMs:

- Case 7 Periodic Microstructure
- Case 8 Descending Horizontal Fiber Strips in One Direction
- Case 9 Descending and Ascending FGMs with Square Fibers
- Case 10 Descending and Symmetric Matrix Structure
- Case 11 Descending Horizontal and Vertical Fiber Strips

The results of the NPH in 1-D cases indicated that the displacement in Case 3, High Density of Microcell Structure, provided better estimation results than in Case 2, Low Density of Microcell Structure. For example, the accuracy of the global deformation with respect to the analytical solution was 98 percent and 95 percent, in Case 3 and in Case 2, respectively. For the Case 5, due to geometric discontinuity, the NPH method did not accurately follow the exact solution – the NPH results were 93 percent compared to the exact solution. However, when the microcell elements were continuously increased and/or decreased in the structure such as Cases 4 and 6, the accuracy of the displacement was presented 99 percent and 98 percent respectively.

For the 2-D Cases, local and global deformation results for the NPH method and the FEA method were compared. The complex FGMs structures are required to have massive number of macro elements and degree-of-freedom (DOF). However, the NPH method used extremely low numbers of macro elements and the NPH results accurately followed the results of the FEA displacement values. In Case 2, the macro elements used 17,956 macro elements for the FEA and 16 elements for the NPH.

The local Von-Mises stresses at the integration points were computed in Cases 10 and 11 based on the global displacement results. For the Case 10, Descending and Symmetric Matrix Structure, the highest stresses occurred at the middle of the structure in the horizontal fiber. The stress contour plots at the microcell level were created to compare the NPH results with the FEA method. The stress contour plots indicated that the NPH stress plots accurately compare with the FEA results. The stress values of the NPH method indicated approximately 29 percent more stress than the FEA estimation.

In Case 11, Descending Horizontal and Vertical Fiber Strips, the highest stress occurred at the right and the bottom of the structure. The stress contours were generated using the same method as Case 10. The results indicated that the maximum stress value was 14 percent higher than the FEA estimation.

## 6.2 Conclusions

A new theory for nonperiodic materials has been developed and verified for basic test cases. In particular, the NPH theory was developed and verified for various complex cases in 1-D and 2-D problems. The NPH local and global deformation results correctly followed the FEA solutions and the Von-Mises stress values were computed. Two major critical factors were discovered in the cases studied. One is the ratio of scale value  $\varepsilon$  (x and y = x/ $\varepsilon$ ). As was shown in the cases studied, a small number  $\varepsilon$  (high density of microcell structure) provided better estimation results than the large number (low density of the microcell structure). In all cases studied, the coefficient value  $\varepsilon$  is not a constant value in the nonperiodic geometry cases. Thus, the coefficient values were varied from 0.018 to 0.180 in the structure. And the changes of the cell size were approximately 10 percent. The other critical factor is the effectiveness of the NPH global displacement method. The accuracy of the local NPH displacement is depended on the global NPH displacement, U(x). Thus, the estimation of the global displacement values is a critical process.

Overall, the NPH program demonstrated that it is a very efficient tool for estimating the local and global displacements as well as computing the microcell the Von-Mises stress levels. The NPH method requires a significantly less model size compare to the conventional FEA method and reduces the computational time. Considering the characteristic of the FGMs materials, which usually have a high density oscillation among the microcell structures and continuously changing

121

volume, the NPH method is an ideal tool for modeling the complex nonperiodic FGMs structures. Therefore, the NPH method is an attractive method for design and estimation of the material behavior under various loading conditions.

# Bibliography

[1] Aboudi, J., Pindera, M. J. and Arnold, S.M., "Higher-order theory for functionally graded materials," Composits Part B 30 (1999) 777-832

[2] Bathe, K. J., "Finite Element Procedures," Prentice-Hall (1996)

[3] Bendsoe, M. P. and Kikuchi, N., "Generating Optimal Topologies in Structural Design using a Homogenization Method," Comp. Methods Appl. Mech. Engreg. 71 (1998) 197-224

[4] Berezovski, Engelbrecht, A., J. and Maugin, G., "Numerical Simulation of Twodimensional Wave Propagation in Functionally Graded Materials," European J. Mechanics A/Solids 22 (2003) 257-265

[5] Budynas, R. G., "Advanced Strength and Applied Stress Analysis," McGaw Hill (1997)

[6] Chen, C. N. and Wellford, L. C. Jr., "Muti-Level Finite Element solution Algorithms Based on Multiplicative and Additive Correction Procedures," Int. J. Numerical Engrg., 28 (1989) 27-41

[7] Cho, J.R. and Ha, D.Y., "Averaging and finite element discretization approaches in the numerical analysis of functionally graded materials," Mater. Sci. Engrg. A 302 (2001) 187-196

[8] Cook, R., Malkus, D. and Plesha M. E., "Concepts and Application of Finite Element Analysis," 3<sup>rd</sup> ed., John Wiley & Sons (1989)

[9] Fish, J., Shek, K., Pandheeradi, M., M. Shephard, S., "Computational Plasticity for Composite Structures based on Methematrical Homogenization: Theory and practice," Comput. Methods Appl. Mech. Engrg., 148 (1997) 53 – 73

[10] Ghosh, S. and Mukhopadhyay, S.N., "A Material Based Finite Elemtent Analysis of Heterogeneous Media involving Dirichlet Tessellations," Comput. Methods Appl. Mech. Eng. 104 (1993) 211-247

[11] Guedes, J. M. and Kikuchi, N., "Preprocessing and Postprocessing for Materials Based on The Homogenization Method with Adaptive Finite Element Methods, Comput. Methods Appl. Mech. Engrg. 83 (1990) 143-198 [12] Hassani, B., "A Direct Method to Derive the Boundary Conditions of the Homogenization Equation for Symmetric Cells," Comm. Int. Numer. Meth. Engrg., 12 (1996)185-196

[13] Hassani, B. and Hinton, E, "Homogenization and Structural Topology Optimization", Springer (1999)

[14] Heroux, M. A. and Thomas, J. W., "A Comparison of FAC and PCG Methods for solving Composite Grid Problems," Comm. Appl. Numerical Methods, 8 (1992) 573-583

[15] Hill, R., "Elastic properties of reinforced properties: some theoretical principles," J. Mech. Phys. Solids 11 (1963) 357-372

[16] Kaminski, M., "Homogenized Properties of Periodic N-Component Composites," Int. J. Engrg. Sci. 38 (2000) 405-427

[17] Kaminski M., "Boundary Element Method Homogenization of the Periodic Linear Elastic Fiber Composites," Engng Analysis with Boundary Elements, 23 (1999) 815-823

[18] Kim, J. G. and Kim, Y. Y., "A New Higher-Order Hybrid-Mixed Curved Beam Element," Int. J. Numer. Meth. Engng 43 (1998) 925-940

[19] Kim, J. H. and Paulino, G. H., "Isoparametric Graded Finite Elements for Nonhomogeneous Isotropic and Orthotropic Materials," ASME J. Appl. Mech, 69 (2002) 502-514

[20] Lee, Y. D. and Erdogan, F., "Residual/Thermal Stresses in FGM and laminated Thermal Barrier Coatings," Int. J. Fract. 69 (1995) 145-165

[21] Markworth, A. J., Ramesh, K. S., and Parks, W.P. Jr., "Modeling Studies Applied to Functionally Graded Materials," J. Mater. Sci. 30 (1995) 2183-2193

[22] Maso, G. D. and Antonio, G.F. D., "Composite Media and Homogenization Theory," Progress in Nonliner Differential Equations and Their Applications (1991)

[23] Matache, A. M. and Schwab, Ch., "Homogenization via p-FEM for problems with Microstructure," Applied Numerical Mathematics 33 (2000) 43-59

[24] Oden, J. T. and Reddy, J.N., "Introduction to the Mathematical Theory of Finite Elements," John Wiley & Sons (1976)

[25] Pindera, M.J., Aboudi, J. and Arnold, S. M., "Thermomechanical Analysis of functionally graded thermal barrier coating with different microstructural scales," J. Am. Ceram. Soc, 81 [6] (1998) 1525-36

[26] Raghavan, P., Moorthy, S., Ghosh, S., Pagano, N. J., "Revisiting the Composit Laminate Problem with an Adaptive Muti-level Computational Model," Composite Science and Technology, 61 (2001) 1017-1040

[27] Reiter T., Dvorak G., Tvergaard, V., "Micromechanical models for graded composite materials," J. Mech. Phys. Solids 45 [8] (1997) 1282-1302

[28] Sadeghi, K. M., "Muti-level Adaptive Finite Element Analysis", Ph.D. Dissertation, University of Southern California, Aug. 1997

[29] Santare, M. H. and Lambros, J., "Use of Graded Finite Elements to Model the Behavior of Nonhomogeneous Materials," ASME J. Appl. Mech., 67 (2000) 819-822

[30] Shkoller, S., "An Approximate Homogenization Scheme for Nonperiodic Materials," Comput. Math. Applic. 33 [4] (1997) 15-34

[31] Sills, L., Eliasi, R. and Berlin, Y., "Modeling of functionally graded materials in dynamic analyses," Composites: Part B 33 (2002) 7-15

[32] Smit, R. J., Brekelmans, W. A. M. and Meijer, H. E. H., "Prediction of the Mechanical Behavior of Nonliner Heterogeneous Systems by Multi-Level Finite Element Modeling," Comput. Methods Appl. Mech. Engrg. 155 (1998) 181-192

[33] Tabiei, A. and Ivanov I., "Computational micro-mechanical model of flexible woven fabric for finite element impact simulation," Int. J. Numer. Meth. Engng., 53 (2002) 1259-1276

[34] Vemaganti, K. and Deshmukh, P., "An Adaptive global-local approach to modeling functionally graded materials," Comput. Methods Appl. Mech. Engrg. 195 (2006) 4230-4243

[35] Wieckowski, Z., "Dual Finite Element Methods in Homogenization for Elastic-Plastic Fibrous Composit Material," Int. J. Plasticity 16 (2000) 199-221

# Appendix A

Nonperiodic Homogenization (NPH) Cell Solution

$$\operatorname{If} \left[ DJBX \left( \overline{\xi} \right) \right]_{4x18} = \begin{bmatrix} [J]^{-1} & 0 \\ 0 & [J]^{-1} \end{bmatrix} \begin{bmatrix} \overline{\partial} & 0 \\ \overline{\partial}\overline{\xi}_{1} & 0 \\ \overline{\partial}\overline{\xi}_{2} & 0 \\ 0 & \overline{\partial}\overline{\xi}_{2} \end{bmatrix} \begin{bmatrix} \beta^{N} \left( \overline{\xi} \right) \right]_{2x18}$$
(A.1)

For the equation (3.19),

$$\begin{cases} u_{1}^{H}(\mathbf{x}) \frac{\partial \beta^{N}(\mathbf{x})}{\partial X_{1}} C_{1}^{N} \\ u_{1}^{H}(\mathbf{x}) \frac{\partial \beta^{N}(\mathbf{x})}{\partial X_{2}} C_{1}^{N} \\ u_{2}^{H}(\mathbf{x}) \frac{\partial \beta^{N}(\mathbf{x})}{\partial X_{1}} C_{2}^{N} \\ u_{2}^{H}(\mathbf{x}) \frac{\partial \beta^{N}(\mathbf{x})}{\partial X_{2}} C_{2}^{N} \\ u_{2}^{H}(\mathbf{x}) \frac{\partial \beta^{N}(\mathbf{x})}{\partial X_{2}} C_{2}^{N} \end{cases} = \begin{bmatrix} u_{1}^{H}(\overline{\xi}) & 0 & 0 & 0 \\ 0 & u_{1}^{H}(\overline{\xi}) & 0 & 0 \\ 0 & 0 & u_{2}^{H}(\overline{\xi}) & 0 \\ 0 & 0 & 0 & u_{2}^{H}(\overline{\xi}) \end{bmatrix} [DJBX(\overline{\xi})]_{4x18} \{C_{i}^{N}\}_{18x1}$$

For the equation (3.20),

(A.3)

For the equation (3.21),

$$\begin{cases}
\frac{\partial u_1^H(\mathbf{x})}{\partial X_1} \beta^N(\mathbf{x}) C_1^N \\
\frac{\partial u_1^H(\mathbf{x})}{\partial X_2} \beta^N(\mathbf{x}) C_1^N \\
\frac{\partial u_2^H(\mathbf{x})}{\partial X_1} \beta^N(\mathbf{x}) C_2^N \\
\frac{\partial u_2^H(\mathbf{x})}{\partial X_2} \beta^N(\mathbf{x}) C_2^N
\end{cases} = \begin{bmatrix} DJBX \begin{bmatrix} \Xi \\ \xi \end{bmatrix}_{4x18} \begin{bmatrix} u_1^{H_1} & 0 \\ 0 & u_2^{H_1} \\ \vdots & \vdots \\ \vdots & \vdots \\ u_1^{H_9} & 0 \\ 0 & u_2^{H_9} \end{bmatrix}_{18x2}
\begin{bmatrix} \beta^N(\overline{\xi}) \end{bmatrix}_{2x18} \{C_i^N\} \quad (A.4)$$

For the equation (3.22),

$$\begin{cases} \chi_{1}^{kl}(\mathbf{x},\mathbf{y}) \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{l}} \frac{\partial \beta^{N}(\mathbf{x})}{\partial X_{1}} \overline{C}_{1}^{N} \\ \chi_{1}^{kl}(\mathbf{x},\mathbf{y}) \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{l}} \frac{\partial \beta^{N}(\mathbf{x})}{\partial X_{2}} \overline{C}_{1}^{N} \\ \chi_{2}^{kl}(\mathbf{x},\mathbf{y}) \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{l}} \frac{\partial \beta^{N}(\mathbf{x})}{\partial X_{1}} \overline{C}_{2}^{N} \\ \chi_{2}^{kl}(\mathbf{x},\mathbf{y}) \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{l}} \frac{\partial \beta^{N}(\mathbf{x})}{\partial X_{2}} \overline{C}_{2}^{N} \\ \end{cases}$$
(A.5)

$$=\begin{bmatrix} \chi_{1}^{kl}(\mathbf{x},\mathbf{y})\frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{l}} & 0 \\ & \chi_{1}^{kl}(\mathbf{x},\mathbf{y})\frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{l}} \\ 0 & \chi_{2}^{kl}(\mathbf{x},\mathbf{y})\frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{l}} \\ 0 & \chi_{2}^{kl}(\mathbf{x},\mathbf{y})\frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{l}} \end{bmatrix} \begin{bmatrix} DJB \end{bmatrix}_{4\times 18} \begin{bmatrix} \overline{C}_{i}^{N} \\ C_{i}^{N} \end{bmatrix}_{18x}$$
(A.6)

Note that, corresponding microcell approximation at the gauss point @ *ij*,  $\mathbf{x} = \mathbf{x}^{ij}$  (for natural coordinate system,  $\overline{\xi}^{ij}$ ), we have the following;

$$\chi_{i}^{kl}(\mathbf{x}, \mathbf{y}) = \psi^{N}(\mathbf{x}, \mathbf{y})\chi_{i}^{kl N}(\mathbf{x})$$

$$= \psi^{N}(\mathbf{x}, \mathbf{y})_{\mathbf{x}=\mathbf{x}^{ij}}\chi_{i}^{kl N}(\mathbf{x})_{\mathbf{x}=\mathbf{x}^{ij}}$$

$$= \psi^{N ij}(\mathbf{y})\chi_{i}^{kl N}(\mathbf{x}^{ij})$$

$$= \chi_{i}^{kl^{ij}}(\mathbf{y})$$
(A.7)

$$\begin{bmatrix} \chi_1^{kl}(\mathbf{x}, \mathbf{y}) \frac{\partial u_k^H(\mathbf{x})}{\partial X_l} \\ \chi_2^{kl}(\mathbf{x}, \mathbf{y}) \frac{\partial u_k^H(\mathbf{x})}{\partial X_l} \end{bmatrix}_{2x1} = \begin{bmatrix} \chi_1^{11}(\mathbf{x}, \mathbf{y}) & \chi_1^{22}(\mathbf{x}, \mathbf{y}) & \chi_1^{12}(\mathbf{x}, \mathbf{y}) \\ \chi_2^{11}(\mathbf{x}, \mathbf{y}) & \chi_2^{22}(\mathbf{x}, \mathbf{y}) & \chi_2^{12}(\mathbf{x}, \mathbf{y}) \end{bmatrix} \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} u_1^H(\mathbf{x})_{X_1} \\ u_1^H(\mathbf{x})_{X_2} \\ u_2^H(\mathbf{x})_{X_1} \\ u_2^H(\mathbf{x})_{X_2} \end{bmatrix}$$

(A.8)

$$= \begin{bmatrix} \chi_1^{11}(\mathbf{x}, \mathbf{y}) & \chi_1^{22}(\mathbf{x}, \mathbf{y}) & \chi_1^{12}(\mathbf{x}, \mathbf{y}) \\ \chi_2^{11}(\mathbf{x}, \mathbf{y}) & \chi_2^{22}(\mathbf{x}, \mathbf{y}) & \chi_2^{12}(\mathbf{x}, \mathbf{y}) \end{bmatrix} \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} DJBX(\vec{\xi}) \\ \vec{\xi} \end{bmatrix} \begin{bmatrix} u_i^H \\ u_i^H \end{bmatrix}_{18x1}$$
(A.9)

$$= \begin{bmatrix} \chi_1^{11^{ij}}(\mathbf{y}) & \chi_1^{22^{ij}}(\mathbf{y}) & \chi_1^{12^{ij}}(\mathbf{y}) \\ \chi_2^{11^{ij}}(\mathbf{y}) & \chi_2^{22^{ij}}(\mathbf{y}) & \chi_2^{12^{ij}}(\mathbf{y}) \end{bmatrix} \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} DJBX \begin{pmatrix} \Xi \\ \xi \end{pmatrix} \end{bmatrix} \begin{bmatrix} u_i^H \\ u_i^H \end{bmatrix}_{18x1}$$
(A.10)

But,

$$\begin{bmatrix} \chi_{1}^{11^{ij}}(\mathbf{y}) & \chi_{2}^{22^{ij}}(\mathbf{y}) & \chi_{1}^{12^{ij}}(\mathbf{y}) \\ \chi_{2}^{11^{ij}}(\mathbf{y}) & \chi_{2}^{22^{ij}}(\mathbf{y}) & \chi_{2}^{12^{ij}}(\mathbf{y}) \end{bmatrix}_{\mathbf{z}^{11^{ij}}} = \begin{bmatrix} \beta^{1ij}(\mathbf{y}) & 0 & & \beta^{9ij}(\mathbf{y}) \\ 0 & \beta^{1ij}(\mathbf{y}) & & & 0 & \beta^{9ij}(\mathbf{y}) \end{bmatrix}_{2\mathbf{z}\mathbf{1}\mathbf{z}\mathbf{1}} \begin{bmatrix} \chi_{1}^{111}(\mathbf{x}^{ij}) & \chi_{1}^{221}(\mathbf{x}^{ij}) & \chi_{1}^{121}(\mathbf{x}^{ij}) \\ \chi_{2}^{111}(\mathbf{x}^{ij}) & \chi_{2}^{221}(\mathbf{x}^{ij}) & \chi_{2}^{121}(\mathbf{x}^{ij}) \\ & & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & & \\ & & & \\ & & &$$

$$= \left[\beta(\xi)\right]_{2x18} \left[\chi_{i}^{11^{ij}N}\right]_{18x3}$$
(A.12)

$$= \left[ \left( \left[ \beta(\xi) \right]_{2x18} \left[ \chi_{i}^{11^{ij} N} \right]_{18x3} \left[ A \right] \left[ DJBX(\overline{\xi}) \right]_{18x1} \left[ u_{i}^{H} \right]_{18x1} \right) \right]_{4x4} \left[ DJB \right]_{4x18} \left\{ \overline{C}_{i} \right\}_{18x1}$$
(A.13)

For the equation (3.23),
$$\begin{bmatrix}
\frac{\partial \chi_{1}^{kl}(\mathbf{x},\mathbf{y})}{\partial X_{1}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{1}} \beta^{N}(\mathbf{x})\overline{C}_{1}^{N} \\
\frac{\partial \chi_{1}^{kl}(\mathbf{x},\mathbf{y})}{\partial X_{2}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{1}} \beta^{N}(\mathbf{x})\overline{C}_{1}^{N} \\
\frac{\partial \chi_{2}^{kl}(\mathbf{x},\mathbf{y})}{\partial X_{1}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{1}} \beta^{N}(\mathbf{x})\overline{C}_{2}^{N} \\
\frac{\partial \chi_{2}^{kl}(\mathbf{x},\mathbf{y})}{\partial X_{2}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{1}} \beta^{N}(\mathbf{x})\overline{C}_{2}^{N}
\end{bmatrix}
=
\begin{bmatrix}
\frac{\partial \chi_{1}^{kl}(\mathbf{x},\mathbf{y})}{\partial X_{2}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{1}} \beta^{N}(\mathbf{x})\overline{C}_{2}^{N} \\
\frac{\partial \chi_{1}^{kl}(\mathbf{x},\mathbf{y})}{\partial X_{2}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{1}} \beta^{N}(\mathbf{x})\overline{C}_{2}^{N}
\end{bmatrix}
=
\begin{bmatrix}
\frac{\partial \chi_{1}^{kl}(\mathbf{x},\mathbf{y})}{\partial X_{2}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{1}} 0 \\
\frac{\partial \chi_{1}^{kl}(\mathbf{x},\mathbf{y})}{\partial X_{2}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{1}} 0 \\
0 & \frac{\partial \chi_{2}^{kl}(\mathbf{x},\mathbf{y})}{\partial X_{2}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{1}} \\
0 & \frac{\partial \chi_{2}^{kl}(\mathbf{x},\mathbf{y})}{\partial X_{2}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{1}} \\
0 & \frac{\partial \chi_{2}^{kl}(\mathbf{x},\mathbf{y})}{\partial X_{2}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{1}} \\
\end{bmatrix}_{4x2}$$
(A.14)

For example,

if 
$$kl = 11$$
 then,  $\frac{\partial \chi_1^{kl}(\mathbf{x}, \mathbf{y})}{\partial X_i} \frac{\partial u_k^H(\mathbf{x})}{\partial X_l}$   $(i = 1, 2)$   

$$\frac{\partial \chi_1^{11}(\mathbf{x}, \mathbf{y})}{\partial X_1} \frac{\partial u_1^H(\mathbf{x})}{\partial X_1} = \left[\frac{\partial \psi^N(\mathbf{x}, \mathbf{y})}{\partial X_1} \chi_1^{11N}(\mathbf{x}) + \psi^N(\mathbf{x}, \mathbf{y}) \frac{\partial \chi_1^{11N}(\mathbf{x})}{\partial X_1}\right] \frac{\partial u_1^H(\mathbf{x})}{\partial X_1}$$

$$\frac{\partial \chi_1^{11}(\mathbf{x}, \mathbf{y})}{\partial X_2} \frac{\partial u_1^H(\mathbf{x})}{\partial X_1} = \left[\frac{\partial \psi^N(\mathbf{x}, \mathbf{y})}{\partial X_2} \chi_1^{11N}(\mathbf{x}) + \psi^N(\mathbf{x}, \mathbf{y}) \frac{\partial \chi_1^{11N}(\mathbf{x})}{\partial X_2}\right] \frac{\partial u_1^H(\mathbf{x})}{\partial X_1}$$

$$\frac{\partial \chi_2^{11}(\mathbf{x}, \mathbf{y})}{\partial X_1} \frac{\partial u_1^H(\mathbf{x})}{\partial X_1} = \left[\frac{\partial \psi^N(\mathbf{x}, \mathbf{y})}{\partial X_1} \chi_2^{11N}(\mathbf{x}) + \psi^N(\mathbf{x}, \mathbf{y}) \frac{\partial \chi_2^{11N}(\mathbf{x})}{\partial X_1}\right] \frac{\partial u_1^H(\mathbf{x})}{\partial X_1}$$

$$\frac{\partial \chi_2^{11}(\mathbf{x}, \mathbf{y})}{\partial X_2} \frac{\partial u_1^H(\mathbf{x})}{\partial X_1} = \left[\frac{\partial \psi^N(\mathbf{x}, \mathbf{y})}{\partial X_2} \chi_2^{11N}(\mathbf{x}) + \psi^N(\mathbf{x}, \mathbf{y}) \frac{\partial \chi_2^{11N}(\mathbf{x})}{\partial X_1}\right] \frac{\partial u_1^H(\mathbf{x})}{\partial X_1}$$

(A.15)

$$= \begin{bmatrix} \left[\frac{\partial \psi^{N}(\mathbf{x}, \mathbf{y})}{\partial h_{2}(\mathbf{x})} \frac{\partial h_{2}(\mathbf{x})}{\partial X_{1}} + \frac{\partial \psi^{N}(\mathbf{x}, \mathbf{y})}{\partial h_{2}(\mathbf{x})} \frac{\partial h_{2}(\mathbf{x})}{\partial X_{1}}\right] \chi_{1}^{11N}(\mathbf{x}) \frac{\partial u_{1}^{H}(\mathbf{x})}{\partial X_{1}} \\ \left[\frac{\partial \psi^{N}(\mathbf{x}, \mathbf{y})}{\partial h_{2}(\mathbf{x})} \frac{\partial h_{2}(\mathbf{x})}{\partial X_{2}} + \frac{\partial \psi^{N}(\mathbf{x}, \mathbf{y})}{\partial h_{2}(\mathbf{x})} \frac{\partial h_{2}(\mathbf{x})}{\partial X_{2}}\right] \chi_{1}^{11N}(\mathbf{x}) \frac{\partial u_{1}^{H}(\mathbf{x})}{\partial X_{1}} \\ \left[\frac{\partial \psi^{N}(\mathbf{x}, \mathbf{y})}{\partial h_{2}(\mathbf{x})} \frac{\partial h_{2}(\mathbf{x})}{\partial X_{1}} + \frac{\partial \psi^{N}(\mathbf{x}, \mathbf{y})}{\partial h_{2}(\mathbf{x})} \frac{\partial h_{2}(\mathbf{x})}{\partial X_{1}}\right] \chi_{2}^{11N}(\mathbf{x}) \frac{\partial u_{1}^{H}(\mathbf{x})}{\partial X_{1}} \\ \left[\frac{\partial \psi^{N}(\mathbf{x}, \mathbf{y})}{\partial h_{2}(\mathbf{x})} \frac{\partial h_{2}(\mathbf{x})}{\partial X_{2}} + \frac{\partial \psi^{N}(\mathbf{x}, \mathbf{y})}{\partial h_{2}(\mathbf{x})} \frac{\partial h_{2}(\mathbf{x})}{\partial X_{2}}\right] \chi_{2}^{11N}(\mathbf{x}) \frac{\partial u_{1}^{H}(\mathbf{x})}{\partial X_{1}} \\ \left[\frac{\partial \psi^{N}(\mathbf{x}, \mathbf{y})}{\partial h_{2}(\mathbf{x})} \frac{\partial h_{2}(\mathbf{x})}{\partial X_{2}} + \frac{\partial \psi^{N}(\mathbf{x}, \mathbf{y})}{\partial h_{2}(\mathbf{x})} \frac{\partial h_{2}(\mathbf{x})}{\partial X_{2}}\right] \chi_{2}^{11N}(\mathbf{x}) \frac{\partial u_{1}^{H}(\mathbf{x})}{\partial X_{1}} \\ \left[\frac{\partial \chi_{1}^{11N}(\mathbf{x})}{\partial h_{2}(\mathbf{x})} \frac{\partial h_{2}(\mathbf{x})}{\partial X_{2}} + \frac{\partial \chi_{1}^{11N}(\mathbf{x})}{\partial h_{2}(\mathbf{x})} \frac{\partial h_{2}(\mathbf{x})}{\partial X_{1}}\right] \\ \left[\frac{\partial \chi_{1}^{11N}(\mathbf{x})}{\partial h_{2}(\mathbf{x})} \frac{\partial h_{2}(\mathbf{x})}{\partial X_{2}} + \frac{\partial \chi_{1}^{11N}(\mathbf{x})}{\partial h_{2}(\mathbf{x})} \frac{\partial h_{2}(\mathbf{x})}{\partial X_{2}}\right] \\ \left[\frac{\partial \chi_{1}^{11N}(\mathbf{x})}{\partial h_{2}(\mathbf{x})} \frac{\partial h_{2}(\mathbf{x})}{\partial X_{2}} + \frac{\partial \chi_{1}^{11N}(\mathbf{x})}{\partial h_{2}(\mathbf{x})} \frac{\partial h_{2}(\mathbf{x})}{\partial X_{2}}}\right] \\ \left[\frac{\partial \chi_{1}^{H}(\mathbf{x})}{\partial h_{2}(\mathbf{x})} \frac{\partial h_{2}(\mathbf{x})}{\partial X_{2}} + \frac{\partial \chi_{1}^{11N}(\mathbf{x})}{\partial h_{2}(\mathbf{x})} \frac{\partial h_{2}(\mathbf{x})}{\partial X_{2}}}\right] \\ \left[\frac{\partial \chi_{1}^{H}(\mathbf{x})}{\partial h_{2}(\mathbf{x})} \frac{\partial \chi_{1}^{H}(\mathbf{x})}{\partial h_{2}(\mathbf{x})} \frac{\partial \chi_{1}^{H}(\mathbf{x})}{\partial h_{2}(\mathbf{x})}}{\partial X_{2}}\right] \\ \left[\frac{\partial \chi_{1}^{H}(\mathbf{x})}{\partial h_{2}(\mathbf{x})} \frac{\partial h_{2}(\mathbf{x})}{\partial X_{2}} + \frac{\partial \chi_{1}^{11N}(\mathbf{x})}{\partial h_{2}(\mathbf{x})} \frac{\partial h_{2}(\mathbf{x})}{\partial X_{1}}}\right] \\ \left[\frac{\partial \chi_{1}^{H}(\mathbf{x})}{\partial h_{2}(\mathbf{x})} \frac{\partial \chi_{1}^{H}(\mathbf{x})}{\partial h_{2}(\mathbf{x})} \frac{\partial \eta_{2}(\mathbf{x})}{\partial X_{2}}}\right] \\ \left[\frac{\partial \chi_{1}^{H}(\mathbf{x})}{\partial h_{2}(\mathbf{x})} \frac{\partial \eta_{2}}{\partial X_{2}} + \frac{\partial \chi_{1}^{H}(\mathbf{x})}{\partial h_{2}(\mathbf{x})} \frac{\partial \eta_{2}(\mathbf{x})}{\partial X_{1}}\right] \\ \left[\frac{\partial \chi_{1}^{H}(\mathbf{x})}{\partial h_{2}(\mathbf{x})} \frac{\partial \eta_{2}}{\partial X_{2}} + \frac{\partial \chi_{1}^{H}(\mathbf{x})}{\partial h_{$$

Have 4x2 matrix form to 4x1 from the equation (A.14), then,

$$\begin{cases} \frac{\partial \chi_{1}^{kl}(\mathbf{x},\mathbf{y})}{\partial X_{1}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{l}} \\ \frac{\partial \chi_{1}^{kl}(\mathbf{x},\mathbf{y})}{\partial X_{2}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{l}} \\ \frac{\partial \chi_{2}^{kl}(\mathbf{x},\mathbf{y})}{\partial X_{1}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{l}} \\ \frac{\partial \chi_{2}^{kl}(\mathbf{x},\mathbf{y})}{\partial X_{2}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{l}} \\ \frac{\partial \chi_{2}^{kl}(\mathbf{x},\mathbf{y})}{\partial X_{2}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{l}} \end{cases} \end{cases}$$

$$= \begin{cases} \frac{\partial \chi_{1}^{11}(\mathbf{x},\mathbf{y})}{\partial X_{1}} \frac{\partial u_{1}^{H}(\mathbf{x})}{\partial X_{1}} + \frac{\partial \chi_{1}^{22}(\mathbf{x},\mathbf{y})}{\partial X_{1}} \frac{\partial u_{2}^{H}(\mathbf{x})}{\partial X_{2}} + \frac{\partial \chi_{1}^{12}(\mathbf{x},\mathbf{y})}{\partial X_{1}} \left( \frac{\partial u_{1}^{H}(\mathbf{x})}{\partial X_{2}} + \frac{\partial u_{2}^{H}(\mathbf{x})}{\partial X_{1}} \right) \\ \frac{\partial \chi_{1}^{11}(\mathbf{x},\mathbf{y})}{\partial X_{2}} \frac{\partial u_{1}^{H}(\mathbf{x})}{\partial X_{1}} + \frac{\partial \chi_{1}^{22}(\mathbf{x},\mathbf{y})}{\partial X_{2}} \frac{\partial u_{2}^{H}(\mathbf{x})}{\partial X_{2}} + \frac{\partial \chi_{1}^{12}(\mathbf{x},\mathbf{y})}{\partial X_{2}} \left( \frac{\partial u_{1}^{H}(\mathbf{x})}{\partial X_{2}} + \frac{\partial u_{2}^{H}(\mathbf{x})}{\partial X_{1}} \right) \\ \frac{\partial \chi_{2}^{11}(\mathbf{x},\mathbf{y})}{\partial X_{1}} \frac{\partial u_{1}^{H}(\mathbf{x})}{\partial X_{1}} + \frac{\partial \chi_{2}^{22}(\mathbf{x},\mathbf{y})}{\partial X_{1}} \frac{\partial u_{2}^{H}(\mathbf{x})}{\partial X_{2}} + \frac{\partial \chi_{1}^{22}(\mathbf{x},\mathbf{y})}{\partial X_{1}} \left( \frac{\partial u_{1}^{H}(\mathbf{x})}{\partial X_{2}} + \frac{\partial u_{2}^{H}(\mathbf{x})}{\partial X_{1}} \right) \\ \frac{\partial \chi_{2}^{11}(\mathbf{x},\mathbf{y})}{\partial X_{2}} \frac{\partial u_{1}^{H}(\mathbf{x})}{\partial X_{1}} + \frac{\partial \chi_{2}^{22}(\mathbf{x},\mathbf{y})}{\partial X_{2}} \frac{\partial u_{2}^{H}(\mathbf{x})}{\partial X_{2}} + \frac{\partial \chi_{2}^{12}(\mathbf{x},\mathbf{y})}{\partial X_{2}} \left( \frac{\partial u_{1}^{H}(\mathbf{x})}{\partial X_{2}} + \frac{\partial u_{2}^{H}(\mathbf{x})}{\partial X_{1}} \right) \\ \frac{\partial \chi_{2}^{11}(\mathbf{x},\mathbf{y})}{\partial X_{2}} \frac{\partial u_{1}^{H}(\mathbf{x})}{\partial X_{1}} + \frac{\partial \chi_{2}^{22}(\mathbf{x},\mathbf{y})}{\partial X_{2}} \frac{\partial u_{2}^{H}(\mathbf{x})}{\partial X_{2}} + \frac{\partial \chi_{2}^{12}(\mathbf{x},\mathbf{y})}{\partial X_{2}} \left( \frac{\partial u_{1}^{H}(\mathbf{x})}{\partial X_{2}} + \frac{\partial u_{2}^{H}(\mathbf{x})}{\partial X_{1}} \right) \\ \frac{\partial \chi_{2}^{11}(\mathbf{x},\mathbf{y})}{\partial X_{2}} \frac{\partial u_{1}^{H}(\mathbf{x})}{\partial X_{1}} + \frac{\partial \chi_{2}^{22}(\mathbf{x},\mathbf{y})}{\partial X_{2}} \frac{\partial u_{2}^{H}(\mathbf{x})}{\partial X_{2}} + \frac{\partial \chi_{2}^{12}(\mathbf{x},\mathbf{y})}{\partial X_{2}} \left( \frac{\partial u_{1}^{H}(\mathbf{x})}{\partial X_{2}} + \frac{\partial u_{2}^{H}(\mathbf{x})}{\partial X_{1}} \right) \\ \frac{\partial \chi_{2}^{11}(\mathbf{x},\mathbf{y})}{\partial X_{2}} \frac{\partial u_{1}^{H}(\mathbf{x})}{\partial X_{1}} + \frac{\partial \chi_{2}^{22}(\mathbf{x},\mathbf{y})}{\partial X_{2}} \frac{\partial u_{2}^{H}(\mathbf{x})}{\partial X_{2}} + \frac{\partial \chi_{2}^{11}(\mathbf{x},\mathbf{y})}{\partial X_{2}} \left( \frac{\partial u_{1}^{H}(\mathbf{x})}{\partial X_{2}} + \frac{\partial u_{2}^{H}(\mathbf{x})}{\partial X_{1}} \right) \\ \frac{\partial \chi_{2}^{11}(\mathbf{x},\mathbf{y})}{\partial X_{2}} \frac{\partial u_{1}^{H}(\mathbf{x})}{\partial X_{2}} + \frac{\partial \chi_{2}^{12}(\mathbf{x},\mathbf{y})}{\partial X_{2}} \left( \frac{\partial u_{1}^{H}(\mathbf{x})}{\partial X_{2}} + \frac{\partial u_{2}^{H}(\mathbf{x})}{\partial X_{1}} \right)$$

(A.17)

$$= \begin{bmatrix} \frac{\partial \chi_{1}^{11}(\mathbf{x},\mathbf{y})}{\partial X_{1}} & \frac{\partial \chi_{1}^{22}(\mathbf{x},\mathbf{y})}{\partial X_{1}} & \frac{\partial \chi_{1}^{12}(\mathbf{x},\mathbf{y})}{\partial X_{1}} \\ \frac{\partial \chi_{1}^{11}(\mathbf{x},\mathbf{y})}{\partial X_{2}} & \frac{\partial \chi_{1}^{22}(\mathbf{x},\mathbf{y})}{\partial X_{2}} & \frac{\partial \chi_{1}^{12}(\mathbf{x},\mathbf{y})}{\partial X_{2}} \\ \frac{\partial \chi_{2}^{11}(\mathbf{x},\mathbf{y})}{\partial X_{1}} & \frac{\partial \chi_{2}^{22}(\mathbf{x},\mathbf{y})}{\partial X_{1}} & \frac{\partial \chi_{2}^{12}(\mathbf{x},\mathbf{y})}{\partial X_{1}} \\ \frac{\partial \chi_{2}^{11}(\mathbf{x},\mathbf{y})}{\partial X_{2}} & \frac{\partial \chi_{2}^{22}(\mathbf{x},\mathbf{y})}{\partial X_{2}} & \frac{\partial \chi_{2}^{12}(\mathbf{x},\mathbf{y})}{\partial X_{1}} \\ \frac{\partial \chi_{2}^{11}(\mathbf{x},\mathbf{y})}{\partial X_{2}} & \frac{\partial \chi_{2}^{22}(\mathbf{x},\mathbf{y})}{\partial X_{2}} & \frac{\partial \chi_{2}^{12}(\mathbf{x},\mathbf{y})}{\partial X_{2}} \end{bmatrix}_{4X3} \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} \frac{\partial u_{1}^{H}(\mathbf{x})}{\partial X_{1}} \\ \frac{\partial u_{1}^{H}(\mathbf{x})}{\partial X_{2}} \\ \frac{\partial u_{2}^{H}(\mathbf{x})}{\partial X_{2}} \\ \frac{\partial u_{2}^{H}(\mathbf{x})}{\partial X_{2}} \end{bmatrix}$$
(A.18)

since 
$$\chi_i^{kl}(\mathbf{x}, \mathbf{y})_{\mathbf{x} = \mathbf{x}^{ij}} = \psi^N(\mathbf{x}, \mathbf{y})_{\mathbf{x} = \mathbf{x}^{ij}} \chi_i^{klN}(\mathbf{x})_{\mathbf{x} = \mathbf{x}^{ij}} = \psi^{Nij}(\mathbf{y}) \chi_i^{klN}(\mathbf{x}^{ij}) = \chi_i^{klij}(\mathbf{y})$$

$$= \begin{bmatrix} \frac{\partial \chi_{1}^{11\,ij}(\mathbf{y})}{\partial X_{1}} & \frac{\partial \chi_{1}^{22\,ij}(\mathbf{y})}{\partial X_{1}} & \frac{\partial \chi_{1}^{12\,ij}(\mathbf{y})}{\partial X_{1}} \\ \frac{\partial \chi_{1}^{11\,ij}(\mathbf{y})}{\partial X_{2}} & \frac{\partial \chi_{1}^{22\,ij}(\mathbf{y})}{\partial X_{2}} & \frac{\partial \chi_{1}^{12\,ij}(\mathbf{y})}{\partial X_{2}} \\ \frac{\partial \chi_{2}^{11\,ij}(\mathbf{y})}{\partial X_{1}} & \frac{\partial \chi_{2}^{22\,ij}(\mathbf{y})}{\partial X_{1}} & \frac{\partial \chi_{2}^{12\,ij}(\mathbf{y})}{\partial X_{1}} \\ \frac{\partial \chi_{2}^{11\,ij}(\mathbf{y})}{\partial X_{2}} & \frac{\partial \chi_{2}^{22\,ij}(\mathbf{y})}{\partial X_{2}} & \frac{\partial \chi_{2}^{12\,ij}(\mathbf{y})}{\partial X_{2}} \\ \frac{\partial \chi_{2}^{11\,ij}(\mathbf{y})}{\partial X_{2}} & \frac{\partial \chi_{2}^{22\,ij}(\mathbf{y})}{\partial X_{2}} & \frac{\partial \chi_{2}^{12\,ij}(\mathbf{y})}{\partial X_{2}} \end{bmatrix}_{4X3} \begin{bmatrix} A \end{bmatrix} DJBX(\overline{\xi}) ] \{ u_{i}^{H} \}$$
(A.19)

Also, note that,

$$\frac{\partial \chi_{1}^{kl \, ij}(\mathbf{y})}{\partial X_{i}} = \frac{\partial \psi^{N \, ij}(\mathbf{y})}{\partial X_{i}} \chi_{i}^{kl \, N}(\mathbf{x}) + \psi^{N \, ij}(\mathbf{y}) \frac{\partial \chi_{1}^{kl \, N}(\mathbf{x})}{\partial X_{i}}$$
$$= \left[ \frac{\partial \psi^{N \, ij}(\mathbf{y})}{\partial h x_{2}(\mathbf{x}^{ij})} \frac{\partial h x_{2}(\mathbf{x}^{ij})}{\partial X_{i}} + \frac{\partial \psi^{N \, ij}(\mathbf{y})}{\partial h y_{2}(\mathbf{x}^{ij})} \frac{\partial h y_{2}(\mathbf{x}^{ij})}{\partial X_{i}} \right] \chi_{i}^{kl \, N}(\mathbf{x})$$
$$+ \psi^{N \, ij}(\mathbf{y}) \left[ \frac{\partial \chi_{1}^{kl \, N}(\mathbf{x})}{\partial h x_{2}(\mathbf{x}^{ij})} \frac{\partial h x_{2}(\mathbf{x}^{ij})}{\partial X_{i}} + \frac{\partial \chi_{1}^{kl \, N}(\mathbf{x})}{\partial h y_{2}(\mathbf{x}^{ij})} \frac{\partial h y_{2}(\mathbf{x}^{ij})}{\partial X_{i}} \right]$$
(A.20)

Thus, the equation (A.19) can be expressed as following,

$$= \begin{bmatrix} \left(\frac{\partial \psi^{Nij}(\mathbf{y})}{\partial h_{2}(\mathbf{x}^{ij})} \frac{\partial h_{2}(\mathbf{x}^{ij})}{\partial X_{1}}\right) \\ \left(\frac{\partial \psi^{Nij}(\mathbf{y})}{\partial h_{2}(\mathbf{x}^{ij})} \frac{\partial h_{2}(\mathbf{x}^{ij})}{\partial X_{2}}\right) \\ \left(\frac{\partial \psi^{Nij}(\mathbf{y})}{\partial h_{2}(\mathbf{x}^{ij})} \frac{\partial h_{2}(\mathbf{x}^{ij})}{$$

(A.21a)

$$+ \begin{pmatrix} \psi^{Nij}(\mathbf{y}) \\ \psi^{Nij}(\mathbf{y}) \\ + \frac{\chi_{1}^{11N}(\mathbf{x}^{ij})}{\partial h_{2}(\mathbf{x}^{ij})} \frac{\partial h_{2}(\mathbf{x}^{ij})}{\partial X_{1}} \\ + \frac{\chi_{1}^{11N}(\mathbf{x}^{ij})}{\partial h_{2}(\mathbf{x}^{ij})} \frac{\partial h_{2}(\mathbf{x}^{ij})}{\partial X_{1}} \end{pmatrix} \psi^{Nij}(\mathbf{y}) \begin{pmatrix} \frac{\chi_{1}^{22N}(\mathbf{x}^{ij})}{\partial h_{2}(\mathbf{x}^{ij})} \frac{\partial h_{2}(\mathbf{x}^{ij})}{\partial X_{1}} \\ + \frac{\chi_{1}^{22N}(\mathbf{x}^{ij})}{\partial h_{2}(\mathbf{x}^{ij})} \frac{\partial h_{2}(\mathbf{x}^{ij})}{\partial X_{1}} \end{pmatrix} \psi^{Nij}(\mathbf{y}) \begin{pmatrix} \frac{\chi_{1}^{22N}(\mathbf{x}^{ij})}{\partial h_{2}(\mathbf{x}^{ij})} \frac{\partial h_{2}(\mathbf{x}^{ij})}{\partial X_{1}} \\ + \frac{\chi_{1}^{12N}(\mathbf{x}^{ij})}{\partial h_{2}(\mathbf{x}^{ij})} \frac{\partial h_{2}(\mathbf{x}^{ij})}{\partial X_{2}} \end{pmatrix} \psi^{Nij}(\mathbf{y}) \begin{pmatrix} \frac{\chi_{1}^{22N}(\mathbf{x}^{ij})}{\partial h_{2}(\mathbf{x}^{ij})} \frac{\partial h_{2}(\mathbf{x}^{ij})}{\partial X_{2}} \\ + \frac{\chi_{1}^{12N}(\mathbf{x}^{ij})}{\partial h_{2}(\mathbf{x}^{ij})} \frac{\partial h_{2}(\mathbf{x}^{ij})}{\partial X_{2}} \end{pmatrix} \psi^{Nij}(\mathbf{y}) \begin{pmatrix} \frac{\chi_{1}^{22N}(\mathbf{x}^{ij})}{\partial h_{2}(\mathbf{x}^{ij})} \frac{\partial h_{2}(\mathbf{x}^{ij})}{\partial X_{2}} \\ + \frac{\chi_{1}^{12N}(\mathbf{x}^{ij})}{\partial h_{2}(\mathbf{x}^{ij})} \frac{\partial h_{2}(\mathbf{x}^{ij})}{\partial X_{2}} \end{pmatrix} \psi^{Nij}(\mathbf{y}) \begin{pmatrix} \frac{\chi_{1}^{22N}(\mathbf{x}^{ij})}{\partial h_{2}(\mathbf{x}^{ij})} \frac{\partial h_{2}(\mathbf{x}^{ij})}{\partial X_{2}} \\ + \frac{\chi_{1}^{12N}(\mathbf{x}^{ij})}{\partial h_{2}(\mathbf{x}^{ij})} \frac{\partial h_{2}(\mathbf{x}^{ij})}{\partial X_{2}} \end{pmatrix} \psi^{Nij}(\mathbf{y}) \begin{pmatrix} \frac{\chi_{1}^{22N}(\mathbf{x}^{ij})}{\partial h_{2}(\mathbf{x}^{ij})} \frac{\partial h_{2}(\mathbf{x}^{ij})}{\partial X_{2}} \\ + \frac{\chi_{1}^{12N}(\mathbf{x}^{ij})}{\partial h_{2}(\mathbf{x}^{ij})} \frac{\partial h_{2}(\mathbf{x}^{ij})}{\partial X_{2}} \end{pmatrix} \psi^{Nij}(\mathbf{y}) \begin{pmatrix} \frac{\chi_{1}^{22N}(\mathbf{x}^{ij})}{\partial h_{2}(\mathbf{x}^{ij})} \frac{\partial h_{2}(\mathbf{x}^{ij})}{\partial X_{2}} \\ + \frac{\chi_{1}^{12N}(\mathbf{x}^{ij})}{\partial h_{2}(\mathbf{x}^{ij})} \frac{\partial h_{2}(\mathbf{x}^{ij})}{\partial X_{2}} \end{pmatrix} \psi^{Nij}(\mathbf{y}) \begin{pmatrix} \frac{\chi_{1}^{22N}(\mathbf{x}^{ij})}{\partial h_{2}(\mathbf{x}^{ij})} \frac{\partial h_{2}(\mathbf{x}^{ij})}{\partial X_{2}} \\ + \frac{\chi_{1}^{22N}(\mathbf{x}^{ij})}{\partial h_{2}(\mathbf{x}^{ij})} \frac{\partial h_{2}(\mathbf{x}^{ij})}{\partial X_{2}} \end{pmatrix} \psi^{Nij}(\mathbf{y}) \begin{pmatrix} \frac{\chi_{1}^{22N}(\mathbf{x}^{ij})}{\partial h_{2}(\mathbf{x}^{ij}} \frac{\partial h_{2}(\mathbf{x}^{ij})}{\partial X_{2}} \\ + \frac{\chi_{1}^{22N}(\mathbf{x}^{ij})}{\partial h_{2}(\mathbf{x}^{ij})} \frac{\partial h_{2}(\mathbf{x}^{ij})}{\partial X_{2}} \end{pmatrix} \psi^{Nij}(\mathbf{y}) \begin{pmatrix} \frac{\chi_{1}^{22N}(\mathbf{x}^{ij})}{\partial h_{2}(\mathbf{x}^{ij}} \frac{\partial h_{2}(\mathbf{x}^{ij})}{\partial X_{1}} \\ + \frac{\chi_{1}^{22N}(\mathbf{x}^{ij})}{\partial h_{2}(\mathbf{x}^{ij})} \frac{\partial h_{2}(\mathbf{x}^{ij})}{\partial X_{1}} \end{pmatrix} \psi^{Nij}(\mathbf{y}) \begin{pmatrix} \frac{\chi_{1}^{22N}(\mathbf{x}^{ij})}{\partial h_{2}(\mathbf{x}^{ij}} \frac{\partial h_{2}(\mathbf{x}^{ij})}}{\partial h$$

And further more,

$$= \begin{bmatrix} \frac{\partial \psi^{1ij}(\mathbf{y})}{\partial hx_2(\mathbf{x}^{ij})} \frac{\partial hx_2(\mathbf{x}^{ij})}{\partial X_1} & 0 & \frac{\partial \psi^{2ij}(\mathbf{y})}{\partial hx_2(\mathbf{x}^{ij})} \frac{\partial hx_2(\mathbf{x}^{ij})}{\partial X_1} & \cdot \\ \frac{\partial \psi^{1ij}(\mathbf{y})}{\partial hx_2(\mathbf{x}^{ij})} \frac{\partial hx_2(\mathbf{x}^{ij})}{\partial X_2} & 0 & \frac{\partial \psi^{2ij}(\mathbf{y})}{\partial hx_2(\mathbf{x}^{ij})} \frac{\partial hx_2(\mathbf{x}^{ij})}{\partial X_2} & \cdot \\ 0 & \frac{\partial \psi^{1ij}(\mathbf{y})}{\partial hx_2(\mathbf{x}^{ij})} \frac{\partial hx_2(\mathbf{x}^{ij})}{\partial X_1} & 0 & \cdot \\ 0 & \frac{\partial \psi^{1ij}(\mathbf{y})}{\partial hx_2(\mathbf{x}^{ij})} \frac{\partial hx_2(\mathbf{x}^{ij})}{\partial X_2} & 0 & \cdot \\ \end{bmatrix}_{4X18}$$

$$+ \begin{bmatrix} \frac{\partial \psi^{1ij}(\mathbf{y})}{\partial h_{2}} \frac{\partial h_{2}(\mathbf{x}^{ij})}{\partial \mathbf{x}_{1}^{11}(\mathbf{x}^{ij})} \frac{\chi_{2}^{12}(\mathbf{x}^{ij})}{\chi_{2}^{22}(\mathbf{x}^{ij})} \frac{\chi_{2}^{12}(\mathbf{x}^{ij})}{\chi_{2}^{21}(\mathbf{x}^{ij})} \\ \vdots & \vdots & \vdots \\ \chi_{1}^{119}(\mathbf{x}^{ij}) \chi_{1}^{22}(\mathbf{x}^{ij}) \chi_{1}^{129}(\mathbf{x}^{ij}) \\ \chi_{2}^{219}(\mathbf{x}^{ij}) \chi_{2}^{229}(\mathbf{x}^{ij}) \chi_{2}^{129}(\mathbf{x}^{ij}) \\ \chi_{2}^{219}(\mathbf{x}^{ij}) \chi_{2}^{229}(\mathbf{x}^{ij}) \chi_{2}^{129}(\mathbf{x}^{ij}) \\ \chi_{2}^{119}(\mathbf{x}^{ij}) \chi_{2}^{229}(\mathbf{x}^{ij}) \chi_{2}^{129}(\mathbf{x}^{ij}) \\ \chi_{2}^{219}(\mathbf{x}^{ij}) \chi_{2}^{229}(\mathbf{x}^{ij}) \frac{\partial h_{2}(\mathbf{x}^{ij})}{\partial \lambda_{1}} & \vdots \\ + \frac{\frac{\partial \psi^{1ij}(\mathbf{y})}{\partial h_{2}(\mathbf{x}^{ij})} \frac{\partial h_{2}(\mathbf{x}^{ij})}{\partial X_{2}} & 0 & \frac{\partial \psi^{2ij}(\mathbf{y})}{\partial h_{2}(\mathbf{x}^{ij})} \frac{\partial h_{2}(\mathbf{x}^{ij})}{\partial X_{2}} & \vdots \\ 0 & \frac{\partial \psi^{1ij}(\mathbf{y})}{\partial h_{2}(\mathbf{x}^{ij})} \frac{\partial h_{2}(\mathbf{x}^{ij})}{\partial X_{2}} & 0 & \vdots \\ 0 & \frac{\partial \psi^{1ij}(\mathbf{y})}{\partial h_{2}(\mathbf{x}^{ij})} \frac{\partial h_{2}(\mathbf{x}^{ij})}{\partial X_{2}} & 0 & \vdots \\ 0 & \frac{\partial \psi^{1ij}(\mathbf{y})}{\partial h_{2}(\mathbf{x}^{ij})} \frac{\chi_{2}^{221}(\mathbf{x}^{ij})}{\partial X_{2}} & 0 & \vdots \\ \frac{\chi_{1}^{111}(\mathbf{x}^{ij}) \chi_{2}^{221}(\mathbf{x}^{ij}) \chi_{2}^{121}(\mathbf{x}^{ij})}{\partial X_{2}} & 0 & \vdots \\ \chi_{1}^{119}(\mathbf{x}^{ij}) \chi_{2}^{229}(\mathbf{x}^{ij}) \chi_{2}^{129}(\mathbf{x}^{ij}) \\ \chi_{2}^{129}(\mathbf{x}^{ij}) \chi_{2}^{229}(\mathbf{x}^{ij}) \chi_{2}^{129}(\mathbf{x}^{ij}) \\ \frac{\chi_{1}^{129}(\mathbf{x}^{ij})}{\partial X_{2}} \frac{\chi_{2}^{229}(\mathbf{x}^{ij}) \chi_{2}^{129}(\mathbf{x}^{ij})}_{\mathbf{1}_{18x3}} & (A.22b) \\ + \begin{bmatrix} \psi^{1ij}(\mathbf{y}) & 0 & 0 & 0 & \psi^{2ij}(\mathbf{y}) & \vdots \\ 0 & \psi^{1ij}(\mathbf{y}) & 0 & 0 & 0 & \vdots \\ 0 & 0 & \psi^{1ij}(\mathbf{y}) & 0 & 0 & \vdots \\ 0 & 0 & \psi^{1ij}(\mathbf{y}) & 0 & 0 & \vdots \\ \end{bmatrix}_{4x16} & \begin{pmatrix} \lambda_{121} & \lambda_{12} & \lambda_{12} & \lambda_{12} & \lambda_{12} \\ \lambda_{12} & \lambda_{12} & \lambda_{12} & \lambda_{12} & \lambda_{12} & \lambda_{12} & \lambda_{13} \\ \chi_{1}^{19}(\mathbf{y}) & \chi_{1}^{29}(\mathbf{x}^{ij}) & \chi_{12}^{29}(\mathbf{x}^{ij}) \\ \chi_{1}^{19}(\mathbf{y}) & 0 & 0 & 0 & \vdots \\ 0 & 0 & \psi^{1ij}(\mathbf{y}) & 0 & 0 & \vdots \\ \end{pmatrix}_{4x36} & \begin{pmatrix} \lambda_{121} & \lambda_{12} & \lambda_{12} & \lambda_{12} & \lambda_{12} & \lambda_{12} & \lambda_{13} \\ \chi_{13} & \chi_{13} & \chi_{13} & \chi_{13} & \chi_{13} \\ \end{pmatrix}_{4x36} & \begin{pmatrix} \lambda_{121} & \lambda_{12} & \lambda_{13} & \lambda_$$

$$\begin{cases} \frac{\partial \chi_{1}^{111}(\mathbf{x}^{ij})}{\partial h_{X_{1}}(\mathbf{x}^{ij})} \frac{\partial h_{X_{2}}(\mathbf{x}^{ij})}{\partial X_{1}} & \frac{\partial \chi_{1}^{221}(\mathbf{x}^{ij})}{\partial h_{X_{2}}(\mathbf{x}^{ij})} \frac{\partial \chi_{1}^{121}(\mathbf{x}^{ij})}{\partial X_{1}} & \frac{\partial \chi_{1}^{121}(\mathbf{x}^{ij})}{\partial h_{X_{1}}(\mathbf{x}^{ij})} \frac{\partial h_{X_{2}}(\mathbf{x}^{ij})}{\partial X_{2}} \\ \frac{\partial \chi_{1}^{111}(\mathbf{x}^{ij})}{\partial h_{X_{2}}(\mathbf{x}^{ij})} \frac{\partial h_{X_{2}}(\mathbf{x}^{ij})}{\partial X_{1}} & \frac{\partial \chi_{2}^{221}(\mathbf{x}^{ij})}{\partial h_{X_{2}}(\mathbf{x}^{ij})} \frac{\partial h_{X_{2}}(\mathbf{x}^{ij})}{\partial X_{1}} & \frac{\partial \chi_{2}^{121}(\mathbf{x}^{ij})}{\partial h_{X_{2}}(\mathbf{x}^{ij})} \frac{\partial h_{X_{2}}(\mathbf{x}^{ij})}{\partial X_{1}} & \frac{\partial \chi_{2}^{121}(\mathbf{x}^{ij})}{\partial h_{X_{2}}(\mathbf{x}^{ij})} \frac{\partial h_{X_{2}}(\mathbf{x}^{ij})}{\partial X_{1}} \\ \frac{\partial \chi_{2}^{111}(\mathbf{x}^{ij})}{\partial h_{X_{2}}(\mathbf{x}^{ij})} \frac{\partial h_{X_{2}}(\mathbf{x}^{ij})}{\partial X_{2}} & \frac{\partial \chi_{2}^{221}(\mathbf{x}^{ij})}{\partial h_{X_{2}}(\mathbf{x}^{ij})} \frac{\partial h_{X_{2}}(\mathbf{x}^{ij})}{\partial X_{2}} & \frac{\partial \chi_{2}^{121}(\mathbf{x}^{ij})}{\partial h_{X_{2}}(\mathbf{x}^{ij})} \frac{\partial h_{X_{2}}(\mathbf{x}^{ij})}{\partial X_{2}} \\ \frac{\partial \chi_{2}^{111}(\mathbf{x}^{ij})}{\partial h_{X_{2}}(\mathbf{x}^{ij})} \frac{\partial h_{X_{2}}(\mathbf{x}^{ij})}{\partial X_{2}} & \frac{\partial \chi_{2}^{221}(\mathbf{x}^{ij})}{\partial h_{X_{2}}(\mathbf{x}^{ij})} \frac{\partial h_{X_{2}}(\mathbf{x}^{ij})}{\partial X_{2}} \\ \frac{\partial \chi_{1}^{111}(\mathbf{x}^{ij})}{\partial h_{Y_{2}}(\mathbf{x}^{ij})} \frac{\partial h_{Y_{2}}(\mathbf{x}^{ij})}{\partial X_{2}} & \frac{\partial \chi_{2}^{221}(\mathbf{x}^{ij})}{\partial h_{X_{2}}(\mathbf{x}^{ij})} \frac{\partial h_{X_{2}}(\mathbf{x}^{ij})}{\partial X_{2}} \\ \frac{\partial \chi_{1}^{111}(\mathbf{x}^{ij})}{\partial h_{Y_{2}}(\mathbf{x}^{ij})} \frac{\partial h_{Y_{2}}(\mathbf{x}^{ij})}{\partial h_{Y_{2}}(\mathbf{x}^{ij})} \frac{\partial h_{Y_{2}}(\mathbf{x}^{ij})}{\partial h_{Y_{2}}(\mathbf{x}^{ij})} \frac{\partial \chi_{1}^{121}(\mathbf{x}^{ij})}{\partial h_{Y_{2}}(\mathbf{x}^{ij})} \frac{\partial h_{Y_{2}}(\mathbf{x}^{ij})}{\partial X_{1}} \\ \frac{\partial \chi_{1}^{111}(\mathbf{x}^{ij})}{\partial h_{Y_{2}}(\mathbf{x}^{ij})} \frac{\partial \chi_{1}^{221}(\mathbf{x}^{ij})}{\partial h_{Y_{2}}(\mathbf{x}^{ij})} \frac{\partial \chi_{2}^{21}(\mathbf{x}^{ij})}{\partial \lambda_{1}} \\ \frac{\partial \chi_{1}^{111}(\mathbf{x}^{ij})}{\partial h_{Y_{2}}(\mathbf{x}^{ij})} \frac{\partial \chi_{2}^{221}(\mathbf{x}^{ij})}{\partial h_{Y_{2}}(\mathbf{x}^{ij})} \frac{\partial \chi_{2}^{21}(\mathbf{x}^{ij})}{\partial h_{Y_{2}}(\mathbf{x}^{ij})}$$

$$\mathbf{y} = (Y_1, Y_2) = (\xi, \eta)$$

$$Y_1 = \xi(\frac{hx_2}{2}), \qquad \frac{\partial\xi}{\partial hx_2(\overline{x}^{ij})} = -\frac{\xi}{hx_2(\overline{x}^{ij})}$$

$$Y_1 = \eta(\frac{hy_2}{2}), \qquad \frac{\partial\eta}{\partial hy_2(\overline{x}^{ij})} = -\frac{\eta}{hy_2(\overline{x}^{ij})}$$
(A.23)

$$\frac{\partial \psi^{1ij}(\mathbf{y})}{\partial hx_2(\mathbf{x}^{ij})} = \frac{\partial \psi^{1ij}(\mathbf{y})}{\partial \xi(\mathbf{x}^{ij})} \frac{\partial \xi(\mathbf{y})}{\partial hx_2(\mathbf{x}^{ij})} = \frac{1}{4} (2\xi - 1)\eta(\eta - 1)(\frac{-\xi}{hx_2(\mathbf{x}^{ij})}) 
\frac{\partial \psi^{2ij}(\mathbf{y})}{\partial hx_2(\mathbf{x}^{ij})} = \frac{1}{4} (2\xi + 1)\eta(\eta - 1)(\frac{-\xi}{hx_2(\mathbf{x}^{ij})}) 
\frac{\partial \psi^{3ij}(\mathbf{y})}{\partial hx_2(\mathbf{x}^{ij})} = \frac{1}{4} (2\xi + 1)\eta(\eta + 1)(\frac{-\xi}{hx_2(\mathbf{x}^{ij})}) 
\frac{\partial \psi^{4ij}(\mathbf{y})}{\partial hx_2(\mathbf{x}^{ij})} = \frac{1}{4} (2\xi - 1)\eta(\eta + 1)(\frac{-\xi}{hx_2(\mathbf{x}^{ij})}) 
\frac{\partial \psi^{5ij}(\mathbf{y})}{\partial hx_2(\mathbf{x}^{ij})} = \frac{1}{2} (-2\xi)\eta(\eta - 1)(\frac{-\xi}{hx_2(\mathbf{x}^{ij})}) 
\frac{\partial \psi^{6ij}(\mathbf{y})}{\partial hx_2(\mathbf{x}^{ij})} = \frac{1}{2} (2\xi + 1)\eta(1 - \eta^2)(\frac{-\xi}{hx_2(\mathbf{x}^{ij})}) 
\frac{\partial \psi^{7ij}(\mathbf{y})}{\partial hx_2(\mathbf{x}^{ij})} = \frac{1}{2} (-2\xi)\eta(\eta + 1)(\frac{-\xi}{hx_2(\mathbf{x}^{ij})}) 
\frac{\partial \psi^{8ij}(\mathbf{y})}{\partial hx_2(\mathbf{x}^{ij})} = \frac{1}{2} (2\xi - 1)\eta(1 - \eta^2)(\frac{-\xi}{hx_2(\mathbf{x}^{ij})}) 
\frac{\partial \psi^{8ij}(\mathbf{y})}{\partial hx_2(\mathbf{x}^{ij})} = \frac{1}{2} (2\xi - 1)\eta(1 - \eta^2)(\frac{-\xi}{hx_2(\mathbf{x}^{ij})}) 
(A.24)$$

$$\frac{\partial \psi^{ii}(\mathbf{y})}{\partial hy_{2}(\mathbf{x}^{ii})} = \frac{\partial \psi^{1ii}(\mathbf{y})}{\partial \eta(\mathbf{x}^{ii})} \frac{\partial \eta(\mathbf{y})}{\partial hy_{2}(\mathbf{x}^{ii})} = \frac{1}{4} \xi(\xi - 1)(2\eta - 1)(\frac{-\eta}{hy_{2}(\mathbf{x}^{ii})})$$

$$\frac{\partial \psi^{2ii}(\mathbf{y})}{\partial hx_{2}(\mathbf{x}^{ii})} = \frac{1}{4} \xi(\xi + 1)(2\eta - 1)(\frac{-\eta}{hy_{2}(\mathbf{x}^{ii})})$$

$$\frac{\partial \psi^{3ii}(\mathbf{y})}{\partial hx_{2}(\mathbf{x}^{ii})} = \frac{1}{4} \xi(\xi + 1)(2\eta + 1)(\frac{-\eta}{hy_{2}(\mathbf{x}^{ii})})$$

$$\frac{\partial \psi^{4ii}(\mathbf{y})}{\partial hx_{2}(\mathbf{x}^{ii})} = \frac{1}{4} \xi(\xi - 1)(2\eta + 1)(\frac{-\eta}{hy_{2}(\mathbf{x}^{ii})})$$

$$\frac{\partial \psi^{5ii}(\mathbf{y})}{\partial hx_{2}(\mathbf{x}^{ii})} = \frac{1}{2} \xi(\xi - 1)(2\eta - 1)(\frac{-\eta}{hy_{2}(\mathbf{x}^{ii})})$$

$$\frac{\partial \psi^{5ii}(\mathbf{y})}{\partial hx_{2}(\mathbf{x}^{ii})} = \frac{1}{2} \xi(\xi + 1)(2\eta - 1)(\frac{-\eta}{hy_{2}(\mathbf{x}^{ii})})$$

$$\frac{\partial \psi^{6ii}(\mathbf{y})}{\partial hx_{2}(\mathbf{x}^{ii})} = \frac{1}{2} \xi(\xi + 1)(-2\eta)(\frac{-\eta}{hy_{2}(\mathbf{x}^{ii})})$$

$$\frac{\partial \psi^{7ii}(\mathbf{y})}{\partial hx_{2}(\mathbf{x}^{ii})} = \frac{1}{2} \xi(\xi - 1)(-2\eta)(\frac{-\eta}{hy_{2}(\mathbf{x}^{ii})})$$

$$\frac{\partial \psi^{8ii}(\mathbf{y})}{\partial hx_{2}(\mathbf{x}^{ii})} = \frac{1}{2} \xi(\xi - 1)(-2\eta)(\frac{-\eta}{hy_{2}(\mathbf{x}^{ii})})$$
(A.25)
$$\frac{\partial \psi^{9ii}(\mathbf{y})}{\partial hx_{2}(\mathbf{x}^{ii})} = (1 - \xi^{2})(-2\eta)(\frac{-\eta}{hy_{2}(\mathbf{x}^{ii})})$$

$$\begin{cases}
\frac{\partial hx_{2}(\mathbf{x}^{ij})}{\partial X_{1}} \\
\frac{\partial hx_{2}(\mathbf{x}^{ij})}{\partial X_{2}} \\
\frac{\partial hy_{2}(\mathbf{x}^{ij})}{\partial X_{2}} \\
\frac{\partial hy_{2}(\mathbf{x}^{ij})}{\partial X_{1}} \\
\frac{\partial hy_{2}(\mathbf{x}^{ij})}{\partial X_{1}} \\
\frac{\partial hy_{2}(\mathbf{x}^{ij})}{\partial X_{2}}
\end{cases} = \begin{cases}
\frac{\partial \psi^{1}(\mathbf{x}^{ij})}{\partial X_{2}} hx_{2}^{1} + \frac{\partial \psi^{2}(\mathbf{x}^{ij})}{\partial X_{2}} hx_{2}^{2} + \frac{\partial \psi^{3}(\mathbf{x}^{ij})}{\partial X_{2}} hx_{2}^{3} + \dots \\
\frac{\partial \psi^{1}(\mathbf{x}^{ij})}{\partial X_{1}} hy_{2}^{1} + \frac{\partial \psi^{2}(\mathbf{x}^{ij})}{\partial X_{1}} hy_{2}^{2} + \frac{\partial \psi^{3}(\mathbf{x}^{ij})}{\partial X_{1}} hy_{2}^{3} + \dots \\
\frac{\partial \psi^{1}(\mathbf{x}^{ij})}{\partial X_{2}} hy_{2}^{1} + \frac{\partial \psi^{2}(\mathbf{x}^{ij})}{\partial X_{2}} hy_{2}^{2} + \frac{\partial \psi^{3}(\mathbf{x}^{ij})}{\partial X_{1}} hy_{2}^{3} + \dots \\
\frac{\partial \psi^{1}(\mathbf{x}^{ij})}{\partial X_{2}} hy_{2}^{1} + \frac{\partial \psi^{2}(\mathbf{x}^{ij})}{\partial X_{2}} hy_{2}^{2} + \frac{\partial \psi^{3}(\mathbf{x}^{ij})}{\partial X_{2}} hy_{2}^{3} + \dots \end{cases}$$
(A.26)

$$= \begin{bmatrix} \frac{\partial}{\partial X_{1}} & 0\\ \frac{\partial}{\partial X_{2}} & 0\\ 0 & \frac{\partial}{\partial X_{1}}\\ 0 & \frac{\partial}{\partial X_{2}} \end{bmatrix} \begin{bmatrix} \beta^{N}(\mathbf{y}^{ij}) \end{bmatrix}_{2X18} \begin{bmatrix} hx_{2}^{1}\\ hy_{2}^{2}\\ hy_{2}^{2}\\ \vdots\\ \vdots\\ \vdots\\ hx_{2}^{0}\\ hy_{2}^{0} \end{bmatrix} = \begin{bmatrix} DJBX(\overline{\xi}) \end{bmatrix}_{4X18} \{hx_{i}^{N}\}$$
(A.27)

Also,

$$\frac{\partial \chi_1^{11N}(\mathbf{x})}{\partial h x_2(\mathbf{x})} = \frac{\chi_1^{11N}(\mathbf{x})_{h x_2(\mathbf{x}) + \Delta h x_2} - \chi_1^{11N}(\mathbf{x})_{h x_2(\mathbf{x})}}{\Delta h x_2}$$
(A.28)

$$\frac{\partial \chi_1^{11N}(\mathbf{x})}{\partial hy_2(\mathbf{x})} = \frac{\chi_1^{11N}(\mathbf{x})_{hy_2(\mathbf{x}) + \Delta hy_2} - \chi_1^{11N}(\mathbf{x})_{hy_2(\mathbf{x})}}{\Delta hy_2}$$
(A.29)

$$\begin{bmatrix} \frac{\partial \chi_{1}^{11\,ij}(\mathbf{y})}{\partial X_{1}} & \frac{\partial \chi_{1}^{22\,ij}(\mathbf{y})}{\partial X_{1}} & \frac{\partial \chi_{1}^{12\,ij}(\mathbf{y})}{\partial X_{1}} \\ \frac{\frac{\partial \chi_{1}^{11\,ij}(\mathbf{y})}{\partial X_{2}} & \frac{\partial \chi_{1}^{22\,ij}(\mathbf{y})}{\partial X_{2}} & \frac{\partial \chi_{1}^{12\,ij}(\mathbf{y})}{\partial X_{2}} \\ \frac{\partial \chi_{2}^{11\,ij}(\mathbf{y})}{\partial X_{1}} & \frac{\partial \chi_{2}^{22\,ij}(\mathbf{y})}{\partial X_{1}} & \frac{\partial \chi_{2}^{12\,ij}(\mathbf{y})}{\partial X_{1}} \\ \frac{\partial \chi_{2}^{11\,ij}(\mathbf{y})}{\partial X_{2}} & \frac{\partial \chi_{2}^{22\,ij}(\mathbf{y})}{\partial X_{2}} & \frac{\partial \chi_{2}^{12\,ij}(\mathbf{y})}{\partial X_{2}} \\ \frac{\partial \chi_{2}^{11\,ij}(\mathbf{y})}{\partial X_{2}} & \frac{\partial \chi_{2}^{22\,ij}(\mathbf{y})}{\partial X_{2}} & \frac{\partial \chi_{2}^{12\,ij}(\mathbf{y})}{\partial X_{2}} \end{bmatrix}_{4X3} \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} DJBX(\overline{\xi}) \end{bmatrix}_{U_{i}}^{U_{i}H} \end{bmatrix}$$

$$= \begin{pmatrix} \left[\frac{\partial \psi^{1ij}(\xi,\eta,\mathbf{x})}{\partial h_{2}(\xi)}\frac{\partial h_{2}(\xi)}{\partial X_{1}}\right]_{4X18} \begin{bmatrix} \chi_{i}^{kl \ N}(\xi) \end{bmatrix}_{18X3} + \left[\frac{\partial \psi^{1ij}(\xi,\eta,\mathbf{x})}{\partial h_{2}(\xi)}\frac{\partial h_{2}(\xi)}{\partial X_{1}}\right]_{4X18} \begin{bmatrix} \chi_{i}^{kl \ N}(\xi) \end{bmatrix}_{18X3} \\ + \left[\psi^{Nij}(\xi,\eta)\right]_{4X36} \begin{bmatrix} \frac{\partial \chi_{1}^{kl \ N}(\xi)}{\partial h_{2}(\xi)}\frac{\partial h_{2}(\xi)}{\partial X_{1}}\frac{\partial h_{2}(\xi)}{\partial X_{1}}\end{bmatrix}_{36X3} \\ + \left[\psi^{Nij}(\xi,\eta)\right]_{4X36} \begin{bmatrix} \frac{\partial \chi_{1}^{kl \ N}(\xi)}{\partial h_{2}(\xi)}\frac{\partial h_{2}(\xi)}{\partial X_{1}}\frac{\partial h_{2}(\xi)}$$

Therefore, the equation A.14 becomes:

$$Equation (A.14) = \begin{cases} \left( \left[ \frac{\partial \psi^{1ij}}{\partial h_{2}(\xi)} \frac{\partial h_{2}(\xi)}{\partial X_{1}} \right]_{4X18} \left[ \chi_{i}^{kl \ N}(\xi) \right]_{18X3} \\ + \left[ \frac{\partial \psi^{1ij}}{\partial h_{2}(\xi)} \frac{\partial h_{2}(\xi)}{\partial X_{1}} \right]_{4X18} \left[ \chi_{i}^{kl \ N}(\xi) \right]_{18X3} \\ + \left[ \psi^{Nij}(\xi,\eta) \right]_{4X36} \left[ \frac{\partial \chi_{1}^{kl \ N}(\xi)}{\partial h_{2}(\xi)} \frac{\partial h_{2}(\xi)}{\partial X_{1}} \right]_{36X3} \\ + \left[ \psi^{Nij}(\xi,\eta) \right]_{4X36} \left[ \frac{\partial \chi_{1}^{kl \ N}(\xi)}{\partial h_{2}(\xi)} \frac{\partial h_{2}(\xi)}{\partial X_{1}} \right]_{36X3} \\ + \left[ \psi^{Nij}(\xi,\eta) \right]_{4X36} \left[ \frac{\partial \chi_{1}^{kl \ N}(\xi)}{\partial h_{2}(\xi)} \frac{\partial h_{2}(\xi)}{\partial X_{1}} \right]_{36X3} \\ + \left[ \psi^{Nij}(\xi,\eta) \right]_{4X36} \left[ \frac{\partial \chi_{1}^{kl \ N}(\xi)}{\partial h_{2}(\xi)} \frac{\partial h_{2}(\xi)}{\partial X_{1}} \right]_{36X3} \\ \times \left[ \beta^{N}(\xi) \right]_{2x18} \left\{ \overline{C}_{i}^{N} \right\}_{18x1} \end{cases}$$

For the equation (3.24),

$$\begin{cases} \chi_{1}^{kl}(\mathbf{x},\mathbf{y}) \frac{\partial^{2} u_{k}^{H}(\mathbf{x})}{\partial X_{1} \partial X_{l}} \beta^{N}(\mathbf{x}) \overline{C}_{1}^{N} \\ \chi_{1}^{kl}(\mathbf{x},\mathbf{y}) \frac{\partial^{2} u_{k}^{H}(\mathbf{x})}{\partial X_{2} \partial X_{l}} \beta^{N}(\mathbf{x}) \overline{C}_{1}^{N} \\ \chi_{2}^{kl}(\mathbf{x},\mathbf{y}) \frac{\partial^{2} u_{k}^{H}(\mathbf{x})}{\partial X_{1} \partial X_{l}} \beta^{N}(\mathbf{x}) \overline{C}_{2}^{N} \\ \chi_{2}^{kl}(\mathbf{x},\mathbf{y}) \frac{\partial^{2} u_{k}^{H}(\mathbf{x})}{\partial X_{2} \partial X_{l}} \beta^{N}(\mathbf{x}) \overline{C}_{2}^{N} \end{cases}$$

$$= \begin{bmatrix} \chi_{1}^{kl}(\mathbf{x},\mathbf{y}) \frac{\partial^{2} u_{k}^{H}(\mathbf{x})}{\partial X_{1} \partial X_{l}} & 0 \\ \chi_{1}^{kl}(\mathbf{x},\mathbf{y}) \frac{\partial^{2} u_{k}^{H}(\mathbf{x})}{\partial X_{2} \partial X_{l}} & 0 \\ 0 & \chi_{2}^{kl}(\mathbf{x},\mathbf{y}) \frac{\partial^{2} u_{k}^{H}(\mathbf{x})}{\partial X_{1} \partial X_{l}} \\ 0 & \chi_{2}^{kl}(\mathbf{x},\mathbf{y}) \frac{\partial^{2} u_{k}^{H}(\mathbf{x})}{\partial X_{2} \partial X_{l}} \end{bmatrix}_{4X2}$$

$$\mathbf{x} \begin{bmatrix} \beta^{1}(\mathbf{x}) & 0 & \beta^{2}(\mathbf{x}) & 0 & \cdots \\ 0 & \beta^{1}(\mathbf{x}) & 0 & \beta^{1}(\mathbf{x}) & \cdots \end{bmatrix}_{2X18} \begin{bmatrix} \overline{C}_{i}^{N} \end{bmatrix}_{18X1} \end{bmatrix}$$

Assemble the matrix form,

$$\begin{cases} \chi_{1}^{kl}(\mathbf{x},\mathbf{y}) \frac{\partial^{2} u_{k}^{H}(\mathbf{x})}{\partial X_{1} \partial X_{l}} \\ \chi_{2}^{kl}(\mathbf{x},\mathbf{y}) \frac{\partial^{2} u_{k}^{H}(\mathbf{x})}{\partial X_{1} \partial X_{l}} \end{cases} \\ = \begin{cases} \chi_{1}^{11}(\mathbf{x},\mathbf{y}) \frac{\partial}{\partial X_{1}} \left( \frac{\partial u_{1}^{H}(\mathbf{x})}{\partial X_{1}} \right) + \chi_{1}^{22}(\mathbf{x},\mathbf{y}) \frac{\partial}{\partial X_{1}} \left( \frac{\partial u_{2}^{H}(\mathbf{x})}{\partial X_{2}} \right) + \chi_{1}^{12}(\mathbf{x},\mathbf{y}) \frac{\partial}{\partial X_{1}} \left( \frac{\partial u_{1}^{H}(\mathbf{x})}{\partial X_{2}} + \frac{\partial u_{2}^{H}(\mathbf{x})}{\partial X_{1}} \right) \\ \chi_{2}^{11}(\mathbf{x},\mathbf{y}) \frac{\partial}{\partial X_{1}} \left( \frac{\partial u_{1}^{H}(\mathbf{x})}{\partial X_{1}} \right) + \chi_{2}^{22}(\mathbf{x},\mathbf{y}) \frac{\partial}{\partial X_{1}} \left( \frac{\partial u_{2}^{H}(\mathbf{x})}{\partial X_{2}} \right) + \chi_{2}^{12}(\mathbf{x},\mathbf{y}) \frac{\partial}{\partial X_{1}} \left( \frac{\partial u_{1}^{H}(\mathbf{x})}{\partial X_{2}} + \frac{\partial u_{2}^{H}(\mathbf{x})}{\partial X_{1}} \right) \end{cases}$$

$$= \begin{bmatrix} \chi_1^{11}(\mathbf{x}, \mathbf{y}) & \chi_1^{22}(\mathbf{x}, \mathbf{y}) & \chi_1^{12}(\mathbf{x}, \mathbf{y}) \\ \chi_2^{11}(\mathbf{x}, \mathbf{y}) & \chi_2^{22}(\mathbf{x}, \mathbf{y}) & \chi_2^{12}(\mathbf{x}, \mathbf{y}) \end{bmatrix} \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial X_1} \left( \frac{\partial u_1^H(\mathbf{x})}{\partial X_1} \right) \\ \frac{\partial}{\partial X_1} \left( \frac{\partial u_1^H(\mathbf{x})}{\partial X_2} \right) \\ \frac{\partial}{\partial X_1} \left( \frac{\partial u_2^H(\mathbf{x})}{\partial X_1} \right) \\ \frac{\partial}{\partial X_1} \left( \frac{\partial u_2^H(\mathbf{x})}{\partial X_2} \right) \end{bmatrix}$$

$$= \left[\beta(\xi)\right]_{2x18} \left[\chi_{i}^{11^{ij}N}\right]_{18x3} \left[A\right] \left\{ \begin{array}{l} \frac{\partial}{\partial X_{1}} \left(\frac{\partial u_{1}^{H}(\mathbf{x})}{\partial X_{1}}\right) \\ \frac{\partial}{\partial X_{1}} \left(\frac{\partial u_{1}^{H}(\mathbf{x})}{\partial X_{2}}\right) \\ \frac{\partial}{\partial X_{1}} \left(\frac{\partial u_{2}^{H}(\mathbf{x})}{\partial X_{1}}\right) \\ \frac{\partial}{\partial X_{1}} \left(\frac{\partial u_{2}^{H}(\mathbf{x})}{\partial X_{2}}\right) \\ \frac{\partial}{\partial X_{1}} \left(\frac{\partial u_{2}^{H}(\mathbf{x})}{\partial X_{2}}\right) \\ \end{array} \right\}$$
(A.33)

Also,

$$\begin{cases} \chi_{1}^{kl}(\mathbf{x},\mathbf{y}) \frac{\partial^{2} u_{k}^{H}(\mathbf{x})}{\partial X_{2} \partial X_{l}} \\ \chi_{2}^{kl}(\mathbf{x},\mathbf{y}) \frac{\partial^{2} u_{k}^{H}(\mathbf{x})}{\partial X_{2} \partial X_{l}} \end{cases} \\ = \begin{cases} \chi_{1}^{11}(\mathbf{x},\mathbf{y}) \frac{\partial}{\partial X_{2}} \left( \frac{\partial u_{1}^{H}(\mathbf{x})}{\partial X_{1}} \right) + \chi_{1}^{22}(\mathbf{x},\mathbf{y}) \frac{\partial}{\partial X_{2}} \left( \frac{\partial u_{2}^{H}(\mathbf{x})}{\partial X_{2}} \right) + \chi_{1}^{12}(\mathbf{x},\mathbf{y}) \frac{\partial}{\partial X_{2}} \left( \frac{\partial u_{1}^{H}(\mathbf{x})}{\partial X_{2}} + \frac{\partial u_{2}^{H}(\mathbf{x})}{\partial X_{1}} \right) \\ \chi_{2}^{11}(\mathbf{x},\mathbf{y}) \frac{\partial}{\partial X_{2}} \left( \frac{\partial u_{1}^{H}(\mathbf{x})}{\partial X_{1}} \right) + \chi_{2}^{22}(\mathbf{x},\mathbf{y}) \frac{\partial}{\partial X_{2}} \left( \frac{\partial u_{2}^{H}(\mathbf{x})}{\partial X_{2}} \right) + \chi_{2}^{12}(\mathbf{x},\mathbf{y}) \frac{\partial}{\partial X_{2}} \left( \frac{\partial u_{1}^{H}(\mathbf{x})}{\partial X_{2}} + \frac{\partial u_{2}^{H}(\mathbf{x})}{\partial X_{1}} \right) \end{cases}$$

(A.34)

$$= \begin{bmatrix} \chi_{1}^{11}(\mathbf{x},\mathbf{y}) & \chi_{1}^{22}(\mathbf{x},\mathbf{y}) & \chi_{1}^{12}(\mathbf{x},\mathbf{y}) \\ \chi_{2}^{11}(\mathbf{x},\mathbf{y}) & \chi_{2}^{22}(\mathbf{x},\mathbf{y}) & \chi_{2}^{12}(\mathbf{x},\mathbf{y}) \end{bmatrix} \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial X_{2}} \left( \frac{\partial u_{1}^{H}(\mathbf{x})}{\partial X_{2}} \right) \\ \frac{\partial}{\partial X_{2}} \left( \frac{\partial u_{1}^{H}(\mathbf{x})}{\partial X_{2}} \right) \\ \frac{\partial}{\partial X_{2}} \left( \frac{\partial u_{2}^{H}(\mathbf{x})}{\partial X_{1}} \right) \\ \frac{\partial}{\partial X_{2}} \left( \frac{\partial u_{2}^{H}(\mathbf{x})}{\partial X_{2}} \right) \end{bmatrix}$$

$$= \left[ \beta(\xi) \right]_{2x18} \left[ \chi_{i}^{11^{ij}N} \right]_{18x3} \left[ A \right] \begin{cases} \frac{\partial}{\partial X_{2}} \left( \frac{\partial u_{1}^{H}(\mathbf{x})}{\partial X_{1}} \right) \\ \frac{\partial}{\partial X_{2}} \left( \frac{\partial u_{1}^{H}(\mathbf{x})}{\partial X_{2}} \right) \\ \frac{\partial}{\partial X_{2}} \left( \frac{\partial u_{2}^{H}(\mathbf{x})}{\partial X_{1}} \right) \\ \frac{\partial}{\partial X_{2}} \left( \frac{\partial u_{2}^{H}(\mathbf{x})}{\partial X_{2}} \right) \end{cases}$$
(A.35)

$$\begin{cases} \frac{\partial}{\partial X_{1}} \left( \frac{\partial u_{1}^{H} (\mathbf{x})}{\partial X_{1}} \right) \\ \frac{\partial}{\partial X_{1}} \left( \frac{\partial u_{1}^{H} (\mathbf{x})}{\partial X_{2}} \right) \\ \frac{\partial}{\partial X_{1}} \left( \frac{\partial u_{2}^{H} (\mathbf{x})}{\partial X_{1}} \right) \\ \frac{\partial}{\partial X_{1}} \left( \frac{\partial u_{2}^{H} (\mathbf{x})}{\partial X_{1}} \right) \\ \frac{\partial}{\partial X_{1}} \left( \frac{\partial u_{2}^{H} (\mathbf{x})}{\partial X_{2}} \right) \end{cases} = \frac{\partial}{\partial X_{1}} \begin{bmatrix} \frac{\partial}{\partial X_{1}} & 0 \\ \frac{\partial}{\partial X_{2}} & 0 \\ 0 & \frac{\partial}{\partial X_{1}} \\ 0 & \frac{\partial}{\partial X_{2}} \end{bmatrix} [\beta(\mathbf{x})]_{2X18} \{u_{i}^{H}\}_{18X1}$$

$$= \frac{\partial}{\partial X_{1}} \begin{bmatrix} [J]^{-1} & 0 \\ 0 & [J]^{-1} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial \overline{\xi}_{1}} & 0 \\ \frac{\partial}{\partial \overline{\xi}_{2}} & 0 \\ 0 & \frac{\partial}{\partial \overline{\xi}_{2}} \end{bmatrix} \begin{bmatrix} \beta^{N}(\underline{\xi}) \end{bmatrix}_{2x18} \begin{cases} u_{i}^{H} \\ u_{i}^{H} \\ u_{i}^{H} \end{cases}$$
(A.36)

$$\begin{cases} \frac{\partial}{\partial X_{2}} \left( \frac{\partial u_{1}^{H}(\mathbf{x})}{\partial X_{1}} \right) \\ \frac{\partial}{\partial X_{2}} \left( \frac{\partial u_{1}^{H}(\mathbf{x})}{\partial X_{2}} \right) \\ \frac{\partial}{\partial X_{2}} \left( \frac{\partial u_{2}^{H}(\mathbf{x})}{\partial X_{1}} \right) \\ \frac{\partial}{\partial X_{2}} \left( \frac{\partial u_{2}^{H}(\mathbf{x})}{\partial X_{1}} \right) \\ \frac{\partial}{\partial X_{2}} \left( \frac{\partial u_{2}^{H}(\mathbf{x})}{\partial X_{2}} \right) \end{cases} = \frac{\partial}{\partial X_{2}} \begin{bmatrix} \frac{\partial}{\partial X_{1}} & 0 \\ \frac{\partial}{\partial X_{2}} & 0 \\ 0 & \frac{\partial}{\partial X_{1}} \\ 0 & \frac{\partial}{\partial X_{2}} \end{bmatrix} \begin{bmatrix} \beta(\mathbf{x}) \end{bmatrix}_{2X18} \{ u_{i}^{H} \}_{18X1} \end{cases}$$

$$= \frac{\partial}{\partial X_{2}} \begin{bmatrix} [J]^{-1} & 0\\ 0 & [J]^{-1} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial \overline{\xi}_{1}} & 0\\ \frac{\partial}{\partial \overline{\xi}_{2}} & 0\\ 0 & \frac{\partial}{\partial \overline{\xi}_{2}} \end{bmatrix} \begin{bmatrix} \beta^{N}(\overline{\underline{\xi}}) \end{bmatrix}_{2 \times 18} \begin{cases} u_{i}^{H} \\ u_{i}^{H} \end{cases}$$
(A.37)

In order to compute the  $\frac{\partial}{\partial X_1}$  and  $\frac{\partial}{\partial X_2}$ , we will get from the following equation:

$$\begin{bmatrix} \frac{\partial}{\partial X_1} & 0\\ \frac{\partial}{\partial X_2} & 0\\ 0 & \frac{\partial}{\partial X_1}\\ 0 & \frac{\partial}{\partial X_2} \end{bmatrix} = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & 0 & 0\\ \Gamma_{21} & \Gamma_{22} & 0 & 0\\ 0 & 0 & \Gamma_{11} & \Gamma_{12}\\ 0 & 0 & \Gamma_{21} & \Gamma_{22} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial \overline{\xi}_1} & 0\\ \frac{\partial}{\partial \overline{\xi}_2} & 0\\ 0 & \frac{\partial}{\partial \overline{\xi}_1} \\ 0 & \frac{\partial}{\partial \overline{\xi}_1} \\ 0 & \frac{\partial}{\partial \overline{\xi}_2} \end{bmatrix}$$

$$= \begin{bmatrix} \Gamma_{11} \frac{\partial}{\partial \overline{\xi}_{1}} + \Gamma_{12} \frac{\partial}{\partial \overline{\xi}_{2}} & 0 \\ \Gamma_{21} \frac{\partial}{\partial \overline{\xi}_{1}} + \Gamma_{22} \frac{\partial}{\partial \overline{\xi}_{2}} & 0 \\ 0 & \Gamma_{11} \frac{\partial}{\partial \overline{\xi}_{1}} + \Gamma_{12} \frac{\partial}{\partial \overline{\xi}_{2}} \\ 0 & \Gamma_{21} \frac{\partial}{\partial \overline{\xi}_{1}} + \Gamma_{22} \frac{\partial}{\partial \overline{\xi}_{2}} \end{bmatrix}$$

(A.38)

$$\therefore \begin{cases} \frac{\partial}{\partial X_1} = \Gamma_{11} \frac{\partial}{\partial \overline{\xi}_1} + \Gamma_{12} \frac{\partial}{\partial \overline{\xi}_2} \\ \frac{\partial}{\partial X_2} = \Gamma_{21} \frac{\partial}{\partial \overline{\xi}_1} + \Gamma_{22} \frac{\partial}{\partial \overline{\xi}_2} \end{cases}$$
(A.39)

where

$$\Gamma_{11} = \frac{1}{J} [J_{22}], \Gamma_{12} = \frac{1}{J} [-J_{12}], \Gamma_{21} = \frac{1}{J} [-J_{21}], \Gamma_{22} = \frac{1}{J} [J_{11}] \text{ and } J = \det[J] \quad (A.40)$$

$$\frac{\partial}{\partial X_{1}} \left[ \begin{bmatrix} J \end{bmatrix}^{-1} & 0 \\ 0 & \begin{bmatrix} J \end{bmatrix}^{-1} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial \overline{\xi}_{1}} & 0 \\ \frac{\partial}{\partial \overline{\xi}_{2}} & 0 \\ 0 & \frac{\partial}{\partial \overline{\xi}_{2}} \end{bmatrix} \begin{bmatrix} \beta^{N} (\underline{\xi}) \end{bmatrix}_{2x18} \left\{ u_{i}^{H} \right\}_{18X1} \\ 0 & \frac{\partial}{\partial \overline{\xi}_{2}} \end{bmatrix} \right]$$

$$= \left( \Gamma_{11} \frac{\partial}{\partial \overline{\xi}_{1}} + \Gamma_{12} \frac{\partial}{\partial \overline{\xi}_{2}} \right) \left[ \begin{bmatrix} J \end{bmatrix}^{-1} & 0 \\ 0 & \begin{bmatrix} J \end{bmatrix}^{-1} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial \overline{\xi}_{1}} & 0 \\ \frac{\partial}{\partial \overline{\xi}_{2}} & 0 \\ 0 & \frac{\partial}{\partial \overline{\xi}_{2}} \end{bmatrix} \begin{bmatrix} \beta^{N} (\underline{\xi}) \end{bmatrix}_{2x18} \left\{ u_{i}^{H} \right\}_{18X1} \quad (A.41)$$

$$= \begin{bmatrix} [J]^{-1} & 0 \\ 0 & [J]^{-1} \end{bmatrix} \begin{bmatrix} \Gamma_{11} \frac{\partial^{2}}{\partial \xi_{1}^{2}} + \Gamma_{12} \frac{\partial^{2}}{\partial \xi_{1}^{2} \partial \xi_{2}} & 0 \\ \Gamma_{11} \frac{\partial^{2}}{\partial \xi_{1}^{2} \partial \xi_{2}} + \Gamma_{12} \frac{\partial^{2}}{\partial \xi_{2}^{2}} & 0 \\ 0 & \Gamma_{11} \frac{\partial^{2}}{\partial \xi_{1}^{2}} + \Gamma_{12} \frac{\partial^{2}}{\partial \xi_{1}^{2} \partial \xi_{2}} \\ 0 & \Gamma_{11} \frac{\partial^{2}}{\partial \xi_{1}^{2} \partial \xi_{2}} + \Gamma_{12} \frac{\partial^{2}}{\partial \xi_{1}^{2} \partial \xi_{2}} \end{bmatrix} \begin{bmatrix} \beta^{N} (\frac{e}{\xi}) \end{bmatrix}_{2x18} \{ u_{i}^{H} \}_{18X1} \\ 0 & \Gamma_{11} \frac{\partial^{2}}{\partial \xi_{1}^{2} \partial \xi_{2}} + \Gamma_{12} \frac{\partial^{2}}{\partial \xi_{2}^{2} \partial \xi_{2}^{2}} \end{bmatrix}$$
(A.42)

$$= \begin{bmatrix} [J]^{-1} & 0 \\ 0 & [J]^{-1} \end{bmatrix}$$

$$\times \begin{pmatrix} \begin{bmatrix} \Gamma_{11} \frac{\partial^2}{\partial \overline{\xi}_1^2} & 0 \\ \Gamma_{11} \frac{\partial^2}{\partial \overline{\xi}_1^2} & 0 \\ 0 & \Gamma_{11} \frac{\partial^2}{\partial \overline{\xi}_1^2} \end{bmatrix}_{2 \times 18} + \begin{bmatrix} \Gamma_{12} \frac{\partial^2}{\partial \overline{\xi}_1^2} & 0 \\ \Gamma_{12} \frac{\partial^2}{\partial \overline{\xi}_2^2} & 0 \\ \Gamma_{12} \frac{\partial^2}{\partial \overline{\xi}_2^2} & 0 \\ 0 & \Gamma_{12} \frac{\partial^2}{\partial \overline{\xi}_1^2} \\ 0 & \Gamma_{12} \frac{\partial^2}{\partial \overline{\xi}_1^2} \end{bmatrix}_{2 \times 18} \begin{bmatrix} \beta^N (\underline{\xi}) \end{bmatrix}_{2 \times 18} + \begin{bmatrix} \Gamma_{12} \frac{\partial^2}{\partial \overline{\xi}_1^2} & 0 \\ \Gamma_{12} \frac{\partial^2}{\partial \overline{\xi}_2^2} & 0 \\ 0 & \Gamma_{12} \frac{\partial^2}{\partial \overline{\xi}_1^2} \\ 0 & \Gamma_{12} \frac{\partial^2}{\partial \overline{\xi}_2^2} \end{bmatrix} \begin{bmatrix} \beta^N (\underline{\xi}) \end{bmatrix}_{2 \times 18} \end{bmatrix} \{ u_i^H \}_{18 \times 1}$$

(A.43)

$$= \left( \begin{bmatrix} J \end{bmatrix}^{-1} \right)_{4X4} \begin{pmatrix} \Gamma_{11} \left( \frac{\partial^2}{\partial \xi_i \partial \xi_j} \right)_{4X2} \begin{bmatrix} \beta^N (\frac{\Xi}{\xi}) \end{bmatrix}_{2x18} \\ + \Gamma_{12} \left( \frac{\partial^2}{\partial \xi_j \partial \xi_i} \right)_{4X2} \begin{bmatrix} \beta^N (\frac{\Xi}{\xi}) \end{bmatrix}_{2x18} \end{pmatrix} \left\{ \mu_i^H \right\}_{18X1}$$
(A.44)

$$\therefore \frac{\partial}{\partial X_{1}} \left[ \begin{bmatrix} J \end{bmatrix}^{-1} & 0 \\ 0 & \begin{bmatrix} J \end{bmatrix}^{-1} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial \overline{\xi}_{1}} & 0 \\ \frac{\partial}{\partial \overline{\xi}_{2}} & 0 \\ 0 & \frac{\partial}{\partial \overline{\xi}_{2}} \\ 0 & \frac{\partial}{\partial \overline{\xi}_{1}} \\ 0 & \frac{\partial}{\partial \overline{\xi}_{2}} \end{bmatrix} \begin{bmatrix} \beta^{N} (\frac{\overline{\xi}}{\underline{\xi}}) \end{bmatrix}_{2x18} \right] \{ u_{i}^{H} \}_{18X1}$$

$$= \left( \begin{bmatrix} J \end{bmatrix}^{-1} \right)_{4X4} \begin{pmatrix} \Gamma_{11} \left( \frac{\partial^2}{\partial \overline{\xi}_i \partial \overline{\xi}_j} \right)_{4X2} \begin{bmatrix} \beta^N \left( \frac{\overline{\xi}}{\underline{\xi}} \right) \end{bmatrix}_{2X18} \\ + \Gamma_{12} \left( \frac{\partial^2}{\partial \overline{\xi}_j \partial \overline{\xi}_i} \right)_{4X2} \begin{bmatrix} \beta^N \left( \frac{\overline{\xi}}{\underline{\xi}} \right) \end{bmatrix}_{2X18} \end{bmatrix} \left\{ u_i^H \right\}_{18X1}$$
(A.45)

$$\therefore \frac{\partial}{\partial X_{2}} \left[ \begin{bmatrix} J \end{bmatrix}^{-1} & 0 \\ 0 & [J]^{-1} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial \overline{\xi}_{1}} & 0 \\ \frac{\partial}{\partial \overline{\xi}_{2}} & 0 \\ 0 & \frac{\partial}{\partial \overline{\xi}_{1}} \\ 0 & \frac{\partial}{\partial \overline{\xi}_{2}} \end{bmatrix} \begin{bmatrix} \beta^{N}(\overline{\underline{\xi}}) \end{bmatrix}_{2x18} \right] \{u_{i}^{H}\}_{18X1}$$

$$= \left( \begin{bmatrix} J \end{bmatrix}^{-1} \right)_{4X4} \left( \frac{\Gamma_{21} \left( \frac{\partial^{2}}{\partial \overline{\xi}_{i} \partial \overline{\xi}_{j}} \right)_{4X2} \left[ \beta^{N}(\overline{\underline{\xi}}) \right]_{2x18}}{+ \Gamma_{22} \left( \frac{\partial^{2}}{\partial \overline{\xi}_{j} \partial \overline{\xi}_{i}} \right)_{4X2} \left[ \beta^{N}(\overline{\underline{\xi}}) \right]_{2x18}} \right] \{u_{i}^{H}\}_{18X1}$$

$$(A.46)$$

$$eq(17) = \begin{bmatrix} \left( \left[ \beta(\xi) \right]_{2x18} \left[ \chi_{i}^{11^{ij}N} \right]_{18x3} \left[ A \right] \right) \left[ \left[ J \right]^{-1} \right]_{4x4} \left\{ \begin{array}{l} \Gamma_{11} \left( \frac{\partial^{2}}{\partial \xi_{i} \partial \xi_{j}} \right)_{4X2} \left[ \beta^{N} \left( \frac{\xi}{2} \right) \right]_{2x18} \\ + \Gamma_{12} \left( \frac{\partial^{2}}{\partial \xi_{j} \partial \xi_{i}} \right)_{4X2} \left[ \beta^{N} \left( \frac{\xi}{2} \right) \right]_{2x18} \right] \left\{ u_{i}^{H} \right\}_{18x1} \\ \left[ \left[ \left[ \beta(\xi) \right]_{2x18} \left[ \chi_{i}^{11^{ij}N} \right]_{18x3} \left[ A \right] \right] \left[ \left[ J \right]^{-1} \right]_{4x4} \left\{ \begin{array}{l} \Gamma_{21} \left( \frac{\partial^{2}}{\partial \xi_{i} \partial \xi_{j}} \right)_{4X2} \left[ \beta^{N} \left( \frac{\xi}{2} \right) \right]_{2x18} \\ + \Gamma_{22} \left( \frac{\partial^{2}}{\partial \xi_{i} \partial \xi_{j}} \right)_{4X2} \left[ \beta^{N} \left( \frac{\xi}{2} \right) \right]_{2x18} \\ + \Gamma_{22} \left( \frac{\partial^{2}}{\partial \xi_{j} \partial \xi_{j}} \right)_{4X2} \left[ \beta^{N} \left( \frac{\xi}{2} \right) \right]_{2x18} \right] \left\{ u_{i}^{H} \right\}_{18x1} \\ + \chi_{\beta}^{N} \left( \frac{\xi}{2} \right)_{2x18} \left\{ C_{i}^{N} \right\}_{18x1} \right]_{4x2} \\ \left\{ \chi_{\beta}^{N} \left( \frac{\xi}{2} \right)_{2x18} \left[ \chi_{\beta}^{N} \left( \frac{\xi}{2} \right) \right]_{2x18} \right\}_{4x2} \left\{ \chi_{\beta}^{N} \left( \frac{\xi}{2} \right) \right\}_{2x18} \left\{ \chi_{\beta}^{H} \right\}_{18x1} \right\}_{4x2} \\ \left\{ \chi_{\beta}^{N} \left( \frac{\xi}{2} \right)_{2x18} \left\{ \zeta_{i}^{N} \right\}_{18x1} \right\}_{4x2} \\ \left\{ \chi_{\beta}^{N} \left( \frac{\xi}{2} \right)_{2x18} \left\{ \zeta_{i}^{N} \right\}_{18x1} \right\}_{4x2} \left\{ \chi_{\beta}^{N} \left( \frac{\xi}{2} \right)_{2x18} \left\{ \zeta_{i}^{N} \right\}_{18x1} \right\}_{4x2} \\ \left\{ \chi_{\beta}^{N} \left( \xi^{N} \right)_{2x18} \left\{ \zeta_{i}^{N} \right\}_{18x1} \right\}_{4x2} \left\{ \chi_{\beta}^{N} \left( \xi^{N} \right)_{2x18} \left\{ \zeta_{i}^{N} \right\}_{18x1} \right\}_{4x2} \right\}_{4x2} \\ \left\{ \chi_{\beta}^{N} \left( \xi^{N} \right)_{2x18} \left\{ \zeta_{i}^{N} \right\}_{18x1} \left\{ \zeta_{i}^{N} \left\{ \zeta_{i}^{N} \right\}_{18x1} \right\}_{4x2} \right\}_{4x2} \\ \left\{ \chi_{\beta}^{N} \left( \xi^{N} \right)_{2x18} \left\{ \zeta_{i}^{N} \left\{ \zeta_{i}^{N} \right\}_{18x1} \right\}_{4x2} \right\}_{4x2} \\ \left\{ \chi_{\beta}^{N} \left\{ \zeta_{i}^{N} \right\}_{18x1} \left\{ \zeta_{i}^{N} \left\{ \zeta_{i}^{N} \right\}_{18x1} \right\}_{4x2} \left\{ \zeta_{i}^{N} \left\{ \zeta_{i}^{N} \right\}_{18x1} \right\}_{4x2} \\ \left\{ \chi_{\beta}^{N} \left\{ \zeta_{i}^{N} \right\}_{18x1} \left\{ \zeta_{i}^{N} \left\{ \zeta_{i}^{N} \right\}_{18x1} \left\{ \zeta_{i}^{N} \left\{ \zeta_{i}^{N} \left\{ \zeta_{i}^{N} \right\}_{18x1} \left\{ \zeta_{i}^{N} \left\{ \zeta_{i}^{N} \left\{ \zeta_{i}^{N} \right\}_{18x1} \left\{ \zeta_{i}^{N} \left$$



For the equation (3.25),

$$\begin{cases}
\frac{\partial \beta^{N}(\mathbf{x})}{\partial X_{1}}u_{1}^{N} \\
\frac{\partial \beta^{N}(\mathbf{x})}{\partial X_{2}}u_{1}^{N} \\
\frac{\partial \beta^{N}(\mathbf{x})}{\partial X_{1}}u_{2}^{N} \\
\frac{\partial \beta^{N}(\mathbf{x})}{\partial X_{1}}u_{2}^{N} \\
\frac{\partial \beta^{N}(\mathbf{x})}{\partial X_{2}}u_{2}^{N}
\end{cases} = \begin{bmatrix}DJBX_{4X18}\{u_{i}^{N}\}_{18X1}\end{bmatrix}$$
(A.48)

For the equation (3.26),

$$\begin{cases}
\frac{\partial \chi_{1}^{kl}(\mathbf{x},\mathbf{y})}{\partial Y_{1}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{l}} \beta^{N}(\mathbf{x})\overline{C}_{1}^{N} \frac{\partial Y_{1}}{\partial X_{1}} \\
\frac{\partial \chi_{1}^{kl}(\mathbf{x},\mathbf{y})}{\partial Y_{2}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{l}} \beta^{N}(\mathbf{x})\overline{C}_{1}^{N} \frac{\partial Y_{2}}{\partial X_{2}} \\
\frac{\partial \chi_{2}^{kl}(\mathbf{x},\mathbf{y})}{\partial Y_{1}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{l}} \beta^{N}(\mathbf{x})\overline{C}_{2}^{N} \frac{\partial Y_{1}}{\partial X_{1}} \\
\frac{\partial \chi_{2}^{kl}(\mathbf{x},\mathbf{y})}{\partial Y_{2}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{l}} \beta^{N}(\mathbf{x})\overline{C}_{2}^{N} \frac{\partial Y_{2}}{\partial X_{2}}
\end{cases}$$
(A.49)

$$= \begin{bmatrix} \frac{\partial \chi_{1}^{kl}(\mathbf{x},\mathbf{y})}{\partial Y_{1}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{l}} \frac{\partial Y_{1}}{\partial X_{1}} & 0 \\ \frac{\partial \chi_{1}^{kl}(\mathbf{x},\mathbf{y})}{\partial Y_{2}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{l}} \frac{\partial Y_{2}}{\partial X_{2}} & 0 \\ 0 & \frac{\partial \chi_{2}^{kl}(\mathbf{x},\mathbf{y})}{\partial Y_{1}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{l}} \frac{\partial Y_{1}}{\partial X_{1}} \\ 0 & \frac{\partial \chi_{2}^{kl}(\mathbf{x},\mathbf{y})}{\partial Y_{2}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{l}} \frac{\partial Y_{1}}{\partial X_{1}} \end{bmatrix} [\beta(\mathbf{x})_{2X18}] [\overline{C}_{i}]_{18X1} \quad (A.50)$$

From the equation A.50, rearrange the matrix from 4x2 to 4x1 format. Then,

$\int \frac{\partial \chi_1^{kl}(\mathbf{x},\mathbf{y})}{\partial \chi_1^{kl}(\mathbf{x},\mathbf{y})}$	$\partial u_k^H(\mathbf{x})$	$\partial Y_1$
$\partial Y_1$	$\partial X_l$	$\partial X_1$
$\partial \chi_1^{kl}(\mathbf{x},\mathbf{y})$	$\partial u_k^H(\mathbf{x})$	$\partial Y_2$
$\partial Y_2$	$\partial X_l$	$\partial X_2$
$\frac{\partial \chi_2^{kl}(\mathbf{x},\mathbf{y})}{\partial \mathbf{x}_2}$	$\frac{\partial u_k^H(\mathbf{x})}{\partial u_k}$	$\partial Y_1$
$\partial Y_1$	$\partial X_l$	$\partial X_1$
$\partial \chi_2^{kl}(\mathbf{x},\mathbf{y})$	$\partial u_k^H(\mathbf{x})$	$\partial Y_2$



Reformat the equation A.51,

$$\begin{bmatrix} \frac{\partial \chi_{1}^{kl}(\mathbf{x},\mathbf{y})}{\partial Y_{1}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{1}} \\ \frac{\partial \chi_{1}^{kl}(\mathbf{x},\mathbf{y})}{\partial Y_{2}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{1}} \\ \frac{\partial \chi_{2}^{kl}(\mathbf{x},\mathbf{y})}{\partial Y_{2}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{1}} \\ \frac{\partial \chi_{2}^{kl}(\mathbf{x},\mathbf{y})}{\partial Y_{1}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{1}} \end{bmatrix} = \begin{bmatrix} \frac{\partial \chi_{1}^{11}(\mathbf{x},\mathbf{y})}{\partial Y_{1}} & \frac{\partial \chi_{1}^{22}(\mathbf{x},\mathbf{y})}{\partial Y_{2}} & \frac{\partial \chi_{1}^{12}(\mathbf{x},\mathbf{y})}{\partial Y_{2}} \\ \frac{\partial \chi_{2}^{kl}(\mathbf{x},\mathbf{y})}{\partial Y_{1}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{1}} \\ \frac{\partial \chi_{2}^{kl}(\mathbf{x},\mathbf{y})}{\partial Y_{2}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{1}} \end{bmatrix} = \begin{bmatrix} \frac{\partial \chi_{1}^{11}(\mathbf{x},\mathbf{y})}{\partial Y_{2}} & \frac{\partial \chi_{1}^{22}(\mathbf{x},\mathbf{y})}{\partial Y_{2}} & \frac{\partial \chi_{1}^{12}(\mathbf{x},\mathbf{y})}{\partial Y_{2}} \\ \frac{\partial \chi_{2}^{kl}(\mathbf{x},\mathbf{y})}{\partial Y_{1}} & \frac{\partial \chi_{2}^{22}(\mathbf{x},\mathbf{y})}{\partial Y_{1}} & \frac{\partial \chi_{2}^{12}(\mathbf{x},\mathbf{y})}{\partial Y_{1}} \\ \frac{\partial \chi_{2}^{l}(\mathbf{x},\mathbf{y})}{\partial Y_{2}} & \frac{\partial \chi_{2}^{22}(\mathbf{x},\mathbf{y})}{\partial Y_{2}} & \frac{\partial \chi_{2}^{22}(\mathbf{x},\mathbf{y})}{\partial Y_{2}} \end{bmatrix} \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} \frac{\partial u_{1}^{H}(\mathbf{x})}{\partial X_{1}} \\ \frac{\partial u_{1}^{H}(\mathbf{x})}{\partial X_{2}} \\ \frac{\partial u_{1}^{H}(\mathbf{x})}{\partial X_{2}} \\ \frac{\partial u_{2}^{H}(\mathbf{x})}{\partial X_{2}} \end{bmatrix} \end{bmatrix}$$

(A.52)

$$= \begin{bmatrix} \frac{\partial \chi_{1}^{11}(\mathbf{x},\mathbf{y})}{\partial Y_{1}} & \frac{\partial \chi_{1}^{22}(\mathbf{x},\mathbf{y})}{\partial Y_{1}} & \frac{\partial \chi_{1}^{12}(\mathbf{x},\mathbf{y})}{\partial Y_{1}} \\ \frac{\partial \chi_{1}^{11}(\mathbf{x},\mathbf{y})}{\partial Y_{2}} & \frac{\partial \chi_{1}^{22}(\mathbf{x},\mathbf{y})}{\partial Y_{2}} & \frac{\partial \chi_{1}^{12}(\mathbf{x},\mathbf{y})}{\partial Y_{2}} \\ \frac{\partial \chi_{2}^{11}(\mathbf{x},\mathbf{y})}{\partial Y_{1}} & \frac{\partial \chi_{2}^{22}(\mathbf{x},\mathbf{y})}{\partial Y_{1}} & \frac{\partial \chi_{2}^{12}(\mathbf{x},\mathbf{y})}{\partial Y_{1}} \\ \frac{\partial \chi_{2}^{11}(\mathbf{x},\mathbf{y})}{\partial Y_{2}} & \frac{\partial \chi_{2}^{22}(\mathbf{x},\mathbf{y})}{\partial Y_{2}} & \frac{\partial \chi_{2}^{12}(\mathbf{x},\mathbf{y})}{\partial Y_{1}} \\ \frac{\partial \chi_{2}^{11}(\mathbf{x},\mathbf{y})}{\partial Y_{2}} & \frac{\partial \chi_{2}^{22}(\mathbf{x},\mathbf{y})}{\partial Y_{2}} & \frac{\partial \chi_{2}^{12}(\mathbf{x},\mathbf{y})}{\partial Y_{2}} \end{bmatrix} \begin{bmatrix} A \llbracket D J B X \rrbracket_{4X18} \{ u_{i}^{H} \}_{18X1} & (A.53) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial \psi^{1}(\mathbf{y})}{\partial Y_{1}} & 0 & \frac{\partial \psi^{2}(\mathbf{y})}{\partial Y_{1}} & 0 & \cdots \\ \frac{\partial \psi^{1}(\mathbf{y})}{\partial Y_{2}} & 0 & \frac{\partial \psi^{2}(\mathbf{y})}{\partial Y_{2}} & 0 & \cdots \\ 0 & \frac{\partial \psi^{1}(\mathbf{y})}{\partial Y_{1}} & 0 & \frac{\partial \psi^{2}(\mathbf{y})}{\partial Y_{1}} & \cdots \\ 0 & \frac{\partial \psi^{1}(\mathbf{y})}{\partial Y_{2}} & 0 & \frac{\partial \psi^{2}(\mathbf{y})}{\partial Y_{2}} & \cdots \\ \end{bmatrix}_{4X18} \begin{bmatrix} \chi_{1}^{111}(\mathbf{x}^{ij}) & \chi_{1}^{221}(\mathbf{x}^{ij}) & \chi_{1}^{121}(\mathbf{x}^{ij}) \\ \chi_{2}^{211}(\mathbf{x}^{ij}) & \chi_{2}^{221}(\mathbf{x}^{ij}) & \chi_{2}^{121}(\mathbf{x}^{ij}) \\ \ddots & \ddots & \ddots \\ \chi_{1}^{119}(\mathbf{x}^{ij}) & \chi_{1}^{229}(\mathbf{x}^{ij}) & \chi_{1}^{129}(\mathbf{x}^{ij}) \\ \chi_{2}^{119}(\mathbf{x}^{ij}) & \chi_{2}^{229}(\mathbf{x}^{ij}) & \chi_{1}^{129}(\mathbf{x}^{ij}) \\ \chi_{2}^{129}(\mathbf{x}^{ij}) & \chi_{2}^{129}(\mathbf{x}^{ij}) \\ \chi_{2}^{119}(\mathbf{x}^{ij}) & \chi_{2}^{229}(\mathbf{x}^{ij}) & \chi_{2}^{129}(\mathbf{x}^{ij}) \\ \chi_{1}^{119}(\mathbf{x}^{ij}) & \chi_{2}^{229}(\mathbf{x}^{ij}) & \chi_{2}^{129}(\mathbf{x}^{ij}) \\ \chi_{1}^{119}(\mathbf{x}^{ij}) & \chi_{2}^{229}(\mathbf{x}^{ij}) & \chi_{1}^{129}(\mathbf{x}^{ij}) \\ \chi_{1}^{119}(\mathbf{x}^{ij}) & \chi_{2}^{210}(\mathbf{x}^{ij}) & \chi_{1}^{129}(\mathbf{x}^{ij}) \\ \chi_{1}^{110}(\mathbf{x}^{ij}) & \chi_{1}^{110}(\mathbf{x}^{ij}) & \chi_{1}^{120}(\mathbf{x}^{ij}) \\ \chi_{1}^{110}(\mathbf{x}^{ij}) & \chi_{1}^{120}(\mathbf{x}^{ij}) & \chi_{1}^{120}(\mathbf{x}^{ij}) \\ \chi_{1}^{110}(\mathbf{x}^{ij}) & \chi_{1}^{120}(\mathbf{x}^{ij}) & \chi_{1}^{120}(\mathbf{x}^{ij}) \\ \chi_{1}^{110}(\mathbf{x}^{ij}) & \chi_{1}^{110}(\mathbf{x}^{ij}) & \chi_{1}^{110}(\mathbf{x}^{ij}) \\ \chi_{1}^{110}(\mathbf{x}^{ij}) & \chi_{1}^{110}(\mathbf{x}^{ij}) & \chi_{1}^{110}(\mathbf{x}^{ij}) \\ \chi_{1}^{110}(\mathbf{x}^{ij}) & \chi_{1}^{110}(\mathbf{x}^{ij}) & \chi_{1}^{110}(\mathbf{x}^{ij}) \\ \chi_{1}^{110}(\mathbf{x}^{ij}) & \chi_{1}^{110}(\mathbf{x}^{ij$$

(A.54)

$$= \left( \begin{bmatrix} \frac{\partial}{\partial Y_{1}} & 0\\ \frac{\partial}{\partial Y_{2}} & 0\\ 0 & \frac{\partial}{\partial Y_{1}}\\ 0 & \frac{\partial}{\partial Y_{2}} \end{bmatrix} [\beta^{N}(\mathbf{y})]_{2X18} [\chi_{1}^{kl N}(\mathbf{x}^{ij})]_{18X3} \right) [A] [DJBX]_{4X18} \{u_{i}^{H}\}_{18X1} \quad (A.55)$$
$$= \left( \begin{bmatrix} DJBY \end{bmatrix}_{4X18} [\chi_{1}^{kl N}(\mathbf{x}^{ij})]_{18X3} \right) [A] [DJBX]_{4X18} \{u_{i}^{H}\}_{18X1} \quad (A.56)$$

$$\begin{cases} \frac{\partial Y_{1}}{\partial X_{1}} = \frac{Y_{1}}{X_{1}} = \frac{2hx_{2} + hx_{1}}{X_{1}} \\ \frac{\partial Y_{2}}{\partial X_{2}} = \frac{Y_{2}}{X_{2}} = \frac{2hy_{2} + hy_{1}}{X_{2}} \end{cases}$$
(A.57)

$$\begin{cases} \frac{\partial \chi_{1}^{kl}(\mathbf{x},\mathbf{y})}{\partial Y_{1}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{l}} \beta^{N}(\mathbf{x})\overline{C}_{1}^{N} \frac{\partial Y_{1}}{\partial X_{1}} \\ \frac{\partial \chi_{1}^{kl}(\mathbf{x},\mathbf{y})}{\partial Y_{2}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{l}} \beta^{N}(\mathbf{x})\overline{C}_{1}^{N} \frac{\partial Y_{2}}{\partial X_{2}} \\ \frac{\partial \chi_{2}^{kl}(\mathbf{x},\mathbf{y})}{\partial Y_{1}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{l}} \beta^{N}(\mathbf{x})\overline{C}_{2}^{N} \frac{\partial Y_{1}}{\partial X_{1}} \\ \frac{\partial \chi_{2}^{kl}(\mathbf{x},\mathbf{y})}{\partial Y_{2}} \frac{\partial u_{k}^{H}(\mathbf{x})}{\partial X_{l}} \beta^{N}(\mathbf{x})\overline{C}_{2}^{N} \frac{\partial Y_{2}}{\partial X_{1}} \end{cases}$$

$$= \left[ \left\{ \left[ \left[ DJBY \right]_{4X18} \left[ \chi_{1}^{kl \ N} \left( \mathbf{x}^{ij} \right) \right]_{8X3} \right] \left[ A \right] \left[ DJBX \right]_{4X18} \left\{ u_{i}^{H} \right\}_{18X1} \right\} \left\{ \frac{\partial Y_{i}}{\partial X_{i}} \right\}_{4X1} \right] \beta^{N} \left( \xi \right)_{2X18} \left\{ \overline{C}_{i}^{N} \right\}_{8X1} \right]$$

$$(A.58)$$